Strong Law of Large Numbers for the Interface in Ballistic Deposition

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Summary. We prove a hydrodynamic limit for ballistic deposition on a multidimensional lattice. In this growth model particles rain down at random and stick to the growing cluster at the first point of contact. The theorem is that if the initial random interface converges to a deterministic macroscopic function, then at later times the height of the scaled interface converges to the viscosity solution of a Hamilton-Jacobi equation. The proof idea is to decompose the interface into the shapes that grow from individual seeds of the initial interface. This decomposition converges to a variational formula that defines viscosity solutions of the macrosopic equation. The technical side of the proof involves subadditive methods and large deviation bounds for related first-passage percolation processes.

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1. Introduction

In a ballistic deposition model particles fall on a surface, find a location where they attach, and become part of the growing cluster. Depending on the rules chosen, the particle may stick to the first point of contact, or it may "roll downhill" and attach itself to the first stable location it encounters. We study the version where particles stick to the first point of contact. A ballistic deposition model is *flux limited* as opposed to *reaction limited* because the rate of growth is limited by the availability of material rather than by the availability of growth sites.

Another qualitative distinction between growth models is the division in *local* and *nonlocal* models. An example of a local model is one where the rate of attaching new particles depends only on the states of some finite number of neighboring sites. A highly nonlocal model is diffusion limited aggregation where the chances of a particle attaching at any particular site are influenced by far-away parts of the growing cluster. Ballistic deposition has a nonlocal aspect because it can build overhangs that extend far sideways to shade parts of the interface that then no longer receive the particle flux. However, the *height* of the interface has local dynamics, and it is this height process that we study in our paper.

The shadowing effect and low atomic mobility (stick to the first point of contact) may play a role in actual deposition processes, so there is physical motivation for these rules (see [16]). We refer the reader to the survey article of Krug and Spohn [18] for a general discussion of growth models and for references to the physics literature.

We prove a *hydrodynamic scaling limit* for a ballistic deposition process on an infinite d-dimensional cubic lattice. Particle deposition happens independently at all sites, governed by exponential waiting times. Once deposited, particles never leave the surface, so in particle system jargon this process is *totally asymmetric*. The correct hydrodynamic scaling shrinks space and speeds up time by the same factor n. The theorem is a law of large numbers: as $n \to \infty$, the height of the randomly evolving surface converges to the solution of a first-order partial differential equation of the Hamilton-Jacobi type. For general treatments of hydrodynamic limits, the reader may consult references [5], [15] and [29].

From a statistical mechanics point of view, our result is a rigorous microscopic derivation of the expected macroscopic theory. The macroscopic growth velocity of the height function ψ is determined by the local slope: $\partial\psi/\partial t = f(\nabla\psi)$. The function f that gives the dependence is the Legendre conjugate of the stationary shape g of a cluster grown from a seed: $f(p) = \sup_x\{x \cdot p + g(x)\}$. As explained in [18], this can be viewed as a Wulff construction for a growing shape.

The essence of the proof is to construct the process so that a supremum of ballistic deposition processes is again a ballistic deposition process. The proof works equally well for discrete time in which case the waiting times of deposited particles are geometric rather than exponential. In fact the Markovian nature of the

dynamics is not needed at all. The proof can be carried out for arbitrary waiting times, but we have not bothered with such generality here. An example can be found in [26].

The paper is organized as follows. Section 2 defines the model more precisely and states the theorems. We discuss briefly viscosity solutions of Hamilton-Jacobi equations, and suggest open problems from the physics literature that may be amenable to rigorous progress. In Section 3 we do the standard graphical construction (see for example [6], [9], [11], or [19]) to show that the process can be rigorously defined on a probability space that supports the Poisson jump time processes. And we construct the coupling that expresses the process as the supremum of the shapes from individual seeds of the initial interface.

In Section 4 we prove exponential large deviation estimates for a first-passage site percolation process. We need these because the lateral (sideways) growth of ballistic deposition from a seed is equivalent to a first-passage percolation problem. The bounds we need are of a standard type, but they are not available in the literature in the right form. We use a result of Talagrand [30] and some ideas from Kesten [14] to derive the inequalities. Grimmett and Kesten [10] have proved such bounds for growth along coordinate axes, but we need to control growth in all space directions. Generalizing the block argument of [10] appeared harder than applying the very general tools of [30].

Theorem 1 is proved in Section 5, and in Section 6 a technical extension of Theorem 1 is proved where the initial seed and initial time are translated as the limit is taken. This is required for the proof of Theorem 2, which is the content of Section 7.

Notational conventions. Frequently used notation is summarized here for the reader's convenience. C, C_1, C_2 are constants whose exact values are immaterial and whose values may change from one inequality to the next. $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$ is the set of nonnegative real numbers, and similarly for $\mathbf{Q}_+$ and $\mathbf{Z}_+$. $\mathbf{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers. *Sites* or points of $\mathbf{Z}^d$ are denoted by u, v, w, z. 0 is the origin of $\mathbf{Z}^d$, and $\mathcal{N}$ is the set of nearest neighbors of the origin in $\mathbf{Z}^d$, in other words, the $2d$ sites at ℓ^1-distance 1 from the origin. To distinguish points of $\mathbf{Z}^{d+1}$ from those of $\mathbf{Z}^d$, we call points of $\mathbf{Z}^{d+1}$ *cells* and denote them by (u, k) where $u \in \mathbf{Z}^d$ and $k \in \mathbf{Z}$. In particular, $(0, 0)$ is the origin of $\mathbf{Z}^d \times \mathbf{Z}_+$. For a real number x, $[x]$ is the maximal integer n subject to $n \leq x$. For $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ the site $[x] = ([x_1], \ldots, [x_d])$, and the ℓ^1 norm is $|x| = |x_1| + \cdots + |x_d|$. For $x, y \in \mathbf{R}$, $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$. $\|g\|_\infty = \sup_x |g(x)|$ for any function g.

$\mathbf{1}_A$ and $\mathbf{1}\{A\}$ denote the indicator random variable of the event A. An $\mathrm{Exp}(\beta)$ random variable X satisfies $P(X > t) = \exp(-\beta t)$ for $t \geq 0$. S_n^1 stands for a sum of n i.i.d. $\mathrm{Exp}(1)$ random variables. The standard notion of stochastic dominance is expressed by $X \preccurlyeq Y$ which is equivalent to $P(X \leq t) \geq P(Y \leq t)$ for all t. θ_s

is a time translation on Poisson point processes that translates points r to $r - s$. In other words, reading $\theta_s\omega$ from time 0 onwards is the same as reading ω from time s onwards. $B_0 \subseteq \mathbf{R}^d$ is the convex compact limiting shape for first-passage site percolation on $\mathbf{Z}^d$ with Exp(1) waiting times.

2. The results

We start with a description of the ballistic deposition process. Fix a positive integer d. Imagine first an arbitrary subset of $\mathbf{Z}^{d+1}$ of occupied *cells*. Cells are simply points of $\mathbf{Z}^{d+1}$ and denoted by (u, h), where $u \in \mathbf{Z}^d$ is a *site* and $h \in \mathbf{Z}$ is a *height*. The *cluster* (subset of occupied cells) grows through deposition events: over each site $u \in \mathbf{Z}^d$, particles rain down randomly at exponential rate 1, independently of all other sites. A particle descending down sticks to the first spot where it touches the existing cluster. We imagine that the particles are exactly the size of a unit cube in $d+1$ dimensions. Over a site u a particle instantaneously drops down from height $h = \infty$ to the highest cell adjacent to the existing cluster, and then this cell becomes occupied and joins the cluster. (Cells are *adjacent* if their ℓ^1 distance is 1.) As the process evolves, a porous structure is generated that grows upward in $d + 1$ dimensions.

We follow the evolution of the top surface of the cluster. For each site u, the *height* σ_u is the maximal h such that cell (u, h) is occupied. We permit the values $\sigma_u = \pm\infty$. $\sigma_u = -\infty$ means that no cell in the column $\{(u, h) : h \in \mathbf{Z}\}$ is occupied. The state of the process is the configuration $\sigma = (\sigma_u : u \in \mathbf{Z}^d)$ of height variables, and the state space is $\mathcal{S} = (\mathbf{Z} \cup \{\pm\infty\})^{\mathbf{Z}^d}$. We write $\sigma(t) = (\sigma_u(t))$ for the height process, where $t \geq 0$ denotes time.

The time evolution of the interface is determined by a collection $\mathcal{T} = \{\mathcal{T}^u : u \in \mathbf{Z}^d\}$ of rate 1 Poisson point processes on the time line $(0, \infty)$. At each epoch of $\mathcal{T}^u$ a particle is deposited above site u. In terms of the height variables $\sigma_u(t)$ a deposition event has a simple expression: Let $\mathcal{N}$ denote the set of $2d$ nearest neighbors of the origin in $\mathbf{Z}^d$. If r is an epoch in Poisson process $\mathcal{T}^u$, the height above site u jumps at time r according to this formula:

$$\sigma_u(r) = \max\bigl[\sigma_u(r-) + 1\,,\, \max\{\sigma_{u+z}(r-) : z \in \mathcal{N}\}\bigr]\,. \tag{2.1}$$

The maximum in (2.1) has the effect that the deposited particle sticks to the highest unoccupied cell above site u that is adjacent to the existing cluster. If no cell above site u is adjacent to the cluster at time $r-$, the right-hand side of (2.1) equals $-\infty$. This means that the deposited particle is lost because it cannot stick to the cluster. In Section 3 we construct this process rigorously starting from an arbitrary initial interface $\sigma(0)$.

The simplest ballistic deposition process starts from a single occupied cell, or *seed*. The prototypical one starts from a seed at the origin. For this process we

reserve the symbol Z and use the symbol σ for the general ballistic deposition process. So at time 0, $Z_0(0) = 0$, and $Z_u(0) = -\infty$ for all sites $u \neq 0$. Otherwise $Z(\cdot)$ evolves as specified above.

Let $\mathcal{B}(t)$ denote the set of sites above which Z has an occupied cell by time t:

$$\mathcal{B}(t) = \{u \in \mathbf{Z}^d : Z_u(t) \geq 0\}. \tag{2.2}$$

Since the original seed is at the origin, $Z_u(t) \geq 0$ is equivalent to $Z_u(t) > -\infty$. The cluster $\mathcal{B}(t)$ is the view of the process $Z(t)$ from above, by projecting $\mathbf{Z}^{d+1}$ onto $\mathbf{Z}^d$ by the map $(u, h) \mapsto u$. The ballistic deposition rules imply that $\mathcal{B}(t)$ grows according to this rule: Each site adjacent to the existing cluster joins independently at rate 1. [Because particles rain down at rate 1 above each site u, and they have a chance of sticking iff some cell above an adjacent site is already occupied.] Thus $\mathcal{B}(t)$ is a familiar growth model, namely *first-passage percolation*, or a version of the Eden growth model. It satisfies a law of large numbers. There is a closed, convex deterministic set $B_0 \subseteq \mathbf{R}^d$ with nonempty interior such that this holds almost surely: for any $\varepsilon > 0$,

$$\big[t(1-\varepsilon)B_0\big] \cap \mathbf{Z}^d \subseteq \mathcal{B}(t) \subseteq \big[t(1+\varepsilon)B_0\big] \cap \mathbf{Z}^d \tag{2.3}$$

for all large enough t. In first-passage percolation literature the random waiting times are usually attached to the edges, but for our $\mathcal{B}(t)$ the waiting times are attached to the sites. In Section 4 we prove some large deviation estimates for $\mathcal{B}(t)$.

Over the set B_0 the scaled ballistic deposition process $t^{-1}Z(t)$ approaches a limiting shape. Let $\operatorname{int} B_0$ denote the topological interior of B_0. We prove in Section 5 this theorem:

Theorem 1. *There exists a bounded positive function g defined on the open convex set* $\operatorname{int} B_0$ *such that this law of large numbers holds: Outside an event of probability zero,*

$$\lim_{n\to\infty} \frac{1}{n} Z_{[nx]}(nt) = tg(x/t) \tag{2.4}$$

for all $x \in \mathbf{R}^d$ and $t > 0$ such that $x/t \in \operatorname{int} B_0$. Furthermore, g is continuous, concave, and invariant under permutations of the coordinate axes and reflections about the origin.

The main theorem is a scaling limit from a general initial interface. Suppose we have a sequence σ^n of ballistic deposition processes. The initial macroscopic interface is given by a function ψ_0 on $\mathbf{R}^d$. We consider three different sets of hypotheses.

Assumption A. ψ_0 is a continuous $[-\infty,+\infty]$-valued function on $\mathbf{R}^d$. The ballistic deposition processes $\sigma^n(t)$ are constructed on a common probability space, and all processes use the same version of the Poisson jump time processes. There is a countable dense subset $Y_0 \subseteq \mathbf{R}^d$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sigma^n_{[ny]}(0) = \psi_0(y) \tag{2.5}$$

almost surely for each $y \in Y_0$, and each $y \in Y_0$ has arbitrarily small closed neighborhoods V such that almost surely

$$\limsup_{n\to\infty} \frac{1}{n} \cdot \max_{u\in nV} \sigma^n_u(0) \le \sup_{y\in V} \psi_0(y)\,. \tag{2.6}$$

Assumption B. ψ_0 is an arbitrary $[-\infty,+\infty]$-valued function on $\mathbf{R}^d$. The ballistic deposition processes $\sigma^n(t)$ are constructed on a common probability space, and all processes use the same version of the Poisson jump time processes. For any fixed $y \in \mathbf{R}^d$, the limit (2.5) holds almost surely, and there are arbitrarily small closed neighborhoods V of y such that (2.6) holds almost surely.

Assumption C. Again ψ_0 is an arbitrary $[-\infty,+\infty]$-valued function on $\mathbf{R}^d$. (2.5) and (2.6) hold in probability in this sense: Let $(\Omega_n, \mathcal{F}_n, P_n)$ be the probability space of the nth process σ^n. For any $y \in \mathbf{R}^d$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n\Big(\big|n^{-1}\sigma^n_{[ny]}(0) - \psi_0(y)\big| \ge \varepsilon \Big) = 0\,, \tag{2.7}$$

and there are arbitrarily small closed neighborhoods V of y such that

$$\lim_{n\to\infty} P_n\Big(\max_{u\in nV} \sigma^n_u(0) \ge n \cdot \sup_{y\in V} \psi_0(y) + n\varepsilon \Big) = 0\,. \tag{2.8}$$

For $x \in \mathbf{R}^d$ and $t > 0$, define

$$\psi(x,t) = \sup_{y\in x+t\cdot \mathrm{int}\, B_0} \left\{ \psi_0(y) + t\, g\left(\frac{x-y}{t}\right) \right\}, \tag{2.9}$$

and $\psi(x,0) = \psi_0(x)$.

Theorem 2. (a) *Strong Laws of Large Numbers: Under Assumption A we have this convergence, simultaneously for all* $x \in \mathbf{R}^d$ *and* $t > 0$*, outside a single exceptional event of probability zero:*

$$\lim_{n\to\infty} \frac{1}{n}\sigma^n_{[nx]}(nt) = \psi(x,t)\,. \tag{2.10}$$

Under Assumption B the limit in (2.10) *holds almost surely for all* (x,t) *at which* ψ *is t-continuous from the right,* $\psi(x,t) = \psi(x,t+)$*.*

(b) *Weak Law of Large Numbers: Under Assumption C the limit in* (2.10) *holds in probability for all* (x,t) *at which* ψ *is t-continuous from the right. In other words, for all such* (x,t)*,*

$$\lim_{n\to\infty} P_n\left(\left|n^{-1}\sigma^n_{[nx]}(nt) - \psi(x,t)\right| \ge \varepsilon \right) = 0 \tag{2.11}$$

for all $\varepsilon > 0$*.*

Remarks and extensions

2.1 The assumptions. Some uniformity assumption such as (2.6) is needed for the result. Consider this example in dimension $d = 1$: The initial interface is

$$\sigma^n_u(0) = \begin{cases} n, & u = 1 \\ 0, & u \neq 1\,. \end{cases}$$

The limit (2.5) is satisfied with $\psi_0(x) \equiv 0$, and then from (2.9) $\psi(x,t) = tg(0)$. But the spike in $\sigma^n_u(0)$ at site $u = 1$ implies that

$$n^{-1}\sigma_0(nt) \ge 1 + n^{-1}Z^1_0(nt) \longrightarrow 1 + tg(0) \qquad \text{as } n \to \infty,$$

where we wrote Z^1 for the process that starts from a seed in cell $(1,0)$.

The point of separating Assumptions A and B is that continuity of ψ_0 guarantees continuity of $\psi(x,t)$ [Lemma 7.1 in Section 7]. Then the limit (2.10) can be proved almost surely, *simultaneously for all* (x,t), even though the hypothesis requires the limit only on a countable dense set Y_0. The limit under Assumption B generalizes Theorem 1. The right-continuity in t at (x,t), namely $\psi(x,t) = \psi(x,t+)$, follows from x-continuity at (x,t), as can be verified from (2.15) below.

The nonlocal shadowing effect restricts the law of large numbers to continuity points of the macroscopic surface. Consider the process Z growing from a seed at $(0,0)$ in dimension $d = 1$. The limit (2.4) cannot possibly hold at $x = t$ because whether site $[nt]$ is shadowed by the top surface varies with the fluctuations of the

right edge of the growing cluster: $Z_{[nt]}(nt) = -\infty$ or ≥ 0 depending on whether $S^1_{[nt]} > nt$ or $\leq nt$, where $S^1_{[nt]}$ is a sum of $[nt]$ i.i.d. Exp(1) random variables and represents the time it takes for the cluster to grow $[nt]$ lattice units to the right.

The proof of Theorem 2 depends on two essential things: a coupling (Lemma 3.3) that expresses the general process as a supremum of processes of the Z type, and the limit in Theorem 1 for the Z process. The coupling is basically a combinatorial property of the paths. As long as pathologies such as jump times accumulating at a point are ruled out, the coupling is independent of the probability distribution on the paths. The limit in Theorem 1 on the other hand comes from an ergodic theorem and is closely tied to the probabilities of the evolution of Z. Thus our proof of Theorem 2 works for any reasonable jump rates under which Theorem 1 can be proved. For example, we could take *quenched* random jump rates where deposition at site u happens at rate α_u, and the rates $\{\alpha_u\}$ are i.i.d. random variables (for examples in other contexts, see [2], [26] and [27]). Another possibility would be to let a sufficiently regular function $\alpha(x)$ on $\mathbf{R}^d$ determine the rates so that deposition at site u for process σ^n happens at rate $\alpha(u/n)$ (for examples, see [1] and [21]).

2.2 Viscosity solutions of Hamilton-Jacobi equations. Consider the Hamilton-Jacobi equation

$$\psi_t - f(\nabla\psi) = 0 \tag{2.12}$$

$$\psi(x,0) = \psi_0(x) \tag{2.13}$$

where f is some continuous function on $\mathbf{R}^d$. A function $\psi(x,t)$ on $\mathbf{R}^d \times \mathbf{R}_+$ that satisfies the initial condition (2.13) is a *viscosity solution* of (2.12)–(2.13) if the following holds for all continuously differentiable functions ϕ on $\mathbf{R}^d \times (0,\infty)$: if $\psi - \phi$ has a local maximum at (x_0, t_0), then

$$\phi_t(x_0,t_0) - f\big(\nabla\phi(x_0,t_0)\big) \leq 0\,,$$

and if $\psi - \phi$ has a local minimum at (x_0, t_0), then

$$\phi_t(x_0,t_0) - f\big(\nabla\phi(x_0,t_0)\big) \geq 0\,.$$

The notion of viscosity solution is due to Crandall and Lions [4]. Properties of viscosity solutions of Hamilton-Jacobi equations are developed in Crandall, Evans and Lions [3], and in the textbook of Evans [8].

Equation (2.9) is known as a *Hopf-Lax formula* in the partial differential equations literature. Let f be the (negative of the) Legendre conjugate of g:

$$f(u) = \sup_{x \in \operatorname{int} B_0} \{u\cdot x + g(x)\}\,, \qquad u \in \mathbf{R}^d\,. \tag{2.14}$$

Since B_0 is a compact set and g is bounded, one can check that f is finite and Lipschitz continuous on all of $\mathbf{R}^d$. Then an argument in [8] (proof of Theorem 3 in Section 10.3.4) shows that $\psi(x,t)$ defined by (2.9) is a viscosity solution of (2.12)–(2.13). By applying results from the p.d.e. literature, we can refine Theorem 1 with a uniqueness statement:

Theorem 3. *Suppose Assumption A is in force, and that additionally the initial macroscopic profile ψ_0 is uniformly continuous on $\mathbf{R}^d$. Fix a finite time horizon $T < \infty$. As in Theorem 1 the strong law of large numbers* (2.10) *is valid. On $\mathbf{R}^d \times [0,T]$ the limit $\psi(x,t)$ is the unique uniformly continuous viscosity solution of the Hamilton-Jacobi equation* (2.12)–(2.13) *whose Hamiltonian $-f$ is defined by* (2.14).

Proof of Theorem 3. The point is that the additional assumption of uniform continuity on ψ_0 permits us to make the uniqueness assertion about ψ. Uniqueness theorems for unbounded viscosity solutions have been proved by Ishii [12], whose Theorem 2.1 states that equation (2.12)–(2.13) with continuous f has a unique uniformly continuous viscosity solution on $\mathbf{R}^d \times [0,T]$. Assume Theorem 2. To prove Theorem 3, we therefore need to check the uniform continuity of ψ defined by (2.9), assuming that ψ_0 is uniformly continuous. Here is an outline of the argument.

First check that formula (2.9) operates like a semigroup: Once ψ has been defined by (2.9), it follows for all $0 < s < t$ that

$$\psi(x,t) = \sup_{y \in x + (t-s)\cdot \operatorname{int} B_0} \left\{ \psi(y,s) + (t-s) g\left(\frac{x-y}{t-s} \right) \right\}. \tag{2.15}$$

This bounds the growth of $\psi(x,t)$ in time. Let $b_0 = \sup\{|y| : y \in B_0\}$. Fix x and $s < t$. (2.15) gives

$$\begin{aligned} 0 &\le \psi(x,t) - \psi(x,s) \\ &\le \sup_{|y'-y''| \le b_0(t-s)} \big| \psi(y',s) - \psi(y'',s) \big| + (t-s)\|g\|_\infty . \end{aligned} \tag{2.16}$$

On the other hand, for a fixed time t the definition (2.9) directly bounds the variation of $\psi(\cdot\,,t)$: For any $x_1, x_2 \in \mathbf{R}^2$ and $t > 0$,

$$\big|\psi(x_1,t) - \psi(x_2,t)\big| \le \sup_{|y'-y''| \le |x_1 - x_2|} \big|\psi_0(y') - \psi_0(y'')\big| . \tag{2.17}$$

Now the uniform continuity of ψ on all of $\mathbf{R}^d \times \mathbf{R}_+$ follows:

$$
\begin{aligned}
&\big|\psi(x_1,t_1) - \psi(x_2,t_2)\big| \\
&\le \big|\psi(x_1,t_1) - \psi(x_2,t_1)\big| + \big|\psi(x_2,t_1) - \psi(x_2,t_2)\big| \\
&\le \sup_{|y'-y''|\le|x_1-x_2|} \big|\psi_0(y') - \psi_0(y'')\big| \\
&\qquad + \sup_{|x'-x''|\le b_0(t_2-t_1)} \big|\psi(x',t_1) - \psi(x'',t_1)\big| + (t_2-t_1)\|g\|_\infty \\
&\qquad\qquad\qquad\qquad [\text{ without loss of generality, assume that } t_1 < t_2 \text{ }] \\
&\le 2\cdot \sup\big\{\big|\psi_0(y') - \psi_0(y'')\big| : |y'-y''| \le |x_1-x_2| \vee b_0|t_2-t_1|\big\} \\
&\qquad + |t_2-t_1|\cdot\|g\|_\infty\,.
\end{aligned}
$$

This proves Theorem 3. $\square$

2.3 Statistical mechanics. From the point of view of statistical mechanics, our paper provides a rigorous derivation of the macroscopic theory that is taken as basic in the physics literature. According to this theory, macroscopically the interface moves under an inclination-dependent growth velocity $f(u)$, and f is the Legendre conjugate of the cluster shape $g(x)$ that grows from a seed. From this basis the physics literature seeks to describe finer properties of the deposition process. The reader is referred to the survey article [18], and to articles [16], [17], and [20].

Here we comment on some properties of the macroscopic objects, and mention open problems suggested by the physics papers. In general, describing f and g is as hard as first-passage percolation, since knowing g would imply knowing the first-passage percolation shape B_0. In dimension $d = 1$ the percolation question becomes trivial, $B_0 = [-1, 1]$, so one may hope to say something about f and g for $d = 1$. The case $d = 1$ is also the focus of the physics literature.

It follows from (2.14) that the velocity $f(u)$ is convex and even $[f(u) = f(-u)]$. Consequently it has a minimum at $u = 0$, $f(0) = g(0)$. Whether this minimum is strict as expected (p. 903 in [17]) is a harder question because that would require showing that g does not have a corner at $x = 0$. These types of questions are open for other interface models too, except in those rare cases where invariant distributions can be used to explicitly compute limiting shapes. For examples, see [23]–[27]. In $d = 1$ (2.14) gives linear asymptotics $f(u) = |u| + g(1) + o(|u|)$ for large slopes $u \to \pm\infty$. The precise nature of the error $o(|u|)$ would be of interest. Equivalently, one wants to know the asymptotics of $g(x) - g(1-)$ as $x \nearrow 1$.

The velocity must increase with dimension, since higher dimension means more neighbors to speed up the growth over a particular site. This is easy to check by coupling the Z-processes for dimensions d and $d+1$ so that the d-dimensional Z-process grows over the hyperplane $\{x_{d+1} = 0\}$ in $\mathbf{Z}^{d+1}$. Without any estimates, this

gives $g_{d+1}(x',0) \geq g_d(x')$ for any $x' \in \mathbf{R}^d$. Consequently, for any $u = (u', u_{d+1}) \in \mathbf{R}^{d+1}$,

$$\begin{aligned} f_{d+1}(u) &\geq \sup_{x' \in \mathbf{R}^d} \{u' \cdot x' + g_{d+1}(x', 0)\} \\ &\geq \sup_{x' \in \mathbf{R}^d} \{u' \cdot x' + g_d(x')\} \\ &= f_d(u')\,. \end{aligned}$$

Mean-field analysis in [17] suggests that $f_d(0)$ grows like $d/\log d$, and simulations appear to show a slow convergence toward mean-field values as $d \to \infty$. Growth at rate $d/\log d$ has been verified for first-passage percolation [13], so these questions can be investigated rigorously.

In a *synchronously updated* ballistic deposition process time is discrete, and a rate parameter $p \in (0,1)$ is fixed. At each time step $t = 1, 2, 3, \dots$, an independent random choice is made at each site: with probability p the height is updated according to equation (2.1), and with probability $1-p$ the height remains the same. The results of our paper apply to the synchronous process without changes. The only difference is that the Poisson point processes $\{\mathcal{T}^w : w \in \mathbf{Z}^d\}$ of jump times are replaced by Bernoulli processes on the discrete time line $\mathbf{N} = \{1, 2, 3, \dots\}$. In these processes an event arrives at each time with probability p, independently of everything else. Analogously with the flat edge result of Durrett and Liggett [7] for first-passage percolation, a *faceting transition* happens in ballistic deposition for large enough p. Interestingly, simulations in [17] suggest that the velocity $f(u)$ is linear for all slopes $u \geq 1$ if p is large enough.

3. Construction and coupling

To construct the ballistic deposition process, start by giving each site $u \in \mathbf{Z}^d$ an independent rate 1 Poisson point process $\mathcal{T}^u$ on the time line $(0, \infty)$. Fix an initial configuration $\sigma(0) = (\sigma_u(0) : u \in \mathbf{Z}^d) \in \mathcal{S} \equiv (\mathbf{Z} \cup \{\pm\infty\})^{\mathbf{Z}^d}$. Informally speaking, the construction of the dynamics goes as follows: If r is an epoch (in other words, a point) of $\mathcal{T}^u$, then at time r height variable σ_u jumps:

$$\sigma_u(r) = \max\bigl[\sigma_u(r-) + 1\,,\, \max\{\sigma_{u+z}(r-) : z \in \mathcal{N}\}\bigr]\,. \tag{3.1}$$

Recall that $\mathcal{N}$ is the set of nearest neighbors of the origin in $\mathbf{Z}^d$.

To make the construction rigorous, we show that there exists a fixed time $t_0 > 0$ and a set of Poisson processes $\{\mathcal{T}^u\}$ of full probability such that, starting with an arbitrary $\sigma(0) \in \mathcal{S}$, the evolution $\sigma(t)$ can be computed for $t \in [0, t_0]$. Since t_0 is independent of $\sigma(0)$, the construction can be repeated, starting with $\sigma(t_0)$, and extended to time interval $[0, 2t_0]$. And so on, to arbitrarily large times.

Given a fixed number $t_0 > 0$ and the Poisson processes $\{\mathcal{T}^u\}$, construct the following random graph with vertex set $\mathbf{Z}^d$: Connect nearest neighbors u and v with an edge if either $\mathcal{T}^u$ or $\mathcal{T}^v$ has an epoch in $[0, t_0]$.

Lemma 3.1. *For small enough fixed $t_0 > 0$, this random graph has no infinite connected components for almost every realization of $\{\mathcal{T}^u\}$.*

Before proving the lemma, let us see how the construction problem is solved. Make these further assumptions on the Poisson point processes, valid for almost every realization:

(3.2a) The $\{\mathcal{T}^u\}$ are such that there are no simultaneous jump attempts.
(3.2b) Each $\mathcal{T}^u$ has finitely may epochs in each bounded time interval.

All sites w that can influence the evolution at site u up to time t_0 are connected to u in the random graph. Since u lies in a finite connected component $\mathcal{C}$, the point process $\cup_{w\in\mathcal{C}}\mathcal{T}^w$ has only finitely many epochs in $[0, t_0]$. Consequently the evolution $\sigma_w(t)$ can be computed for $w \in \mathcal{C}$ and $t \in [0, t_0]$ from rule (3.1), by considering the finitely many epochs in their temporal order. This procedure is repeated for all connected components.

Proof of Lemma 3.1. By translation invariance, it suffices to show that the origin is almost surely connected to only finitely many vertices. $\{u_0, u_1, \dots, u_n\}$ is a self-avoiding path of length n in the random graph if $u_i \neq u_j$ for $i \neq j$ and if there is an edge between u_i and u_{i+1} for each i. If the origin is connected to a site u with $|u| \geq L$, there is a self-avoiding path of length $\geq L$ starting at the origin. The probability that a self-avoiding path of length $2n-1$ starts at the origin is at most

$$(2d)^{2n-1}\big(1 - e^{-2t_0}\big)^n .$$

The factor $(2d)^{2n-1}$ is an upper bound on the number of such paths. If $0 = u_0, u_1, \dots, u_{2n-1}$ is such a path, the n edges (u_0, u_1), (u_2, u_3), $\dots$, (u_{2n-2}, u_{2n-1}) are present independently of each other, and each with probability $1-e^{-2t_0}$ [at least one of $\mathcal{T}^{u_{2i}}$ and $\mathcal{T}^{u_{2i+1}}$ must have an epoch in $[0, t_0]$, and each $\mathcal{T}^{u_j}$ has rate 1]. Pick t_0 small enough so that $(2d)^2\big(1 - e^{-2t_0}\big) < 1$. Then by Borel-Cantelli, self-avoiding paths from the origin have a finite upper bound on their length, almost surely. $\square$

This approach to the construction of a particle system is due to Harris [11]. Our presentation followed [6].

Let $(\Omega, \mathcal{F}, P)$ denote the probability space whose sample point ω represents a realization of the Poisson processes $\mathcal{T} = \{\mathcal{T}^u\}$. We constructed the random path $\sigma(\cdot) = (\sigma_u(t) : u \in \mathbf{Z}^d, t \geq 0)$ as a function of the given initial state $\sigma(0)$ and a sample point ω. Since the Poisson processes are Markovian, the process $\sigma(\cdot)$ is a time-homogeneous Markov process. When the initial interface $\sigma(0)$ is random, the underlying probability space is constructed so that $\sigma(0)$ and $\mathcal{T}$ are independent.

Formula (3.1) preserves ordering, so we get the following monotonicity lemma, whose proof is left to the reader:

Lemma 3.2. *Suppose σ and ρ are ballistic deposition processes constructed on a common probability space so that they use the same version $\{\mathcal{T}^u\}$ of the Poisson processes. Assume that at time 0, $\sigma_u(0) \geq \rho_u(0)$ for all $u \in \mathbf{Z}^d$, almost surely. Then almost surely $\sigma_u(t) \geq \rho_u(t)$ for all $u \in \mathbf{Z}^d$ and $t \geq 0$.*

A less obvious property of the construction is the following, which forms the basis of our approach to the hydrodynamic limit.

Lemma 3.3. *Suppose the ballistic deposition process σ and a countable family of ballistic deposition processes $\{\zeta^i : i \in \mathcal{I}\}$ are constructed on a common probability space so that they all use the same version $\{\mathcal{T}^u\}$ of the Poisson processes. Assume that at time 0, almost surely,*

$$\sigma_u(0) = \sup_{i\in\mathcal{I}} \zeta^i_u(0) \quad \textit{for all } u \in \mathbf{Z}^d. \tag{3.3}$$

Then almost surely

$$\sigma_u(t) = \sup_{i\in\mathcal{I}} \zeta^i_u(t) \quad \textit{for all } u \in \mathbf{Z}^d \textit{ and } t \geq 0. \tag{3.4}$$

Proof. First apply Lemma 3.2 with $\rho = \zeta^i$ to get

$$\sigma_u(t) \geq \sup_{i\in\mathcal{I}} \zeta^i_u(t) \quad \text{for all } u \in \mathbf{Z}^d \text{ and } t \geq 0. \tag{3.5}$$

Thus we need to show that for all sites u and times t, there is some index i such that $\sigma_u(t) = \zeta^i_u(t)$.

Pick and fix a realization $\{\mathcal{T}^u\}$ that satisfies assumptions (3.2) and for which the conclusion of Lemma 3.1 holds. We first show that, for any processes σ and $\{\zeta^i : i \in \mathcal{I}\}$ that satisfy the hypotheses, (3.4) holds for $t \in (0, t_0]$ where $t_0 > 0$ is the number chosen in Lemma 3.1. To do so for a fixed site u^0, let $\mathcal{C} \subseteq \mathbf{Z}^d$ be the finite connected component of u^0 in the (random) graph constructed for Lemma 3.1. The evolutions of all the processes on the sites of $\mathcal{C}$ are determined by the finitely many Poisson points in $\cup_{w\in\mathcal{C}}\mathcal{T}^w \cap (0, t_0]$. We can now prove (3.4) up to time t_0 by checking that it holds right after each jump.

So suppose $r \in (0, t_0]$ is a jump time in $\mathcal{T}^v$ for some site $v \in \mathcal{C}$ so that (3.1) happens for $u = v$. Assume by induction that (3.4) holds for $t < r$, for all $u \in \mathcal{C}$. Since the variables σ_u and ζ^i_u are $\mathbf{Z}$-valued, this means that at each time $t < r$ the supremum in (3.4) is actually achieved at some $i \in \mathcal{I}$. Depending on how the jump (3.1) for $u = v$ is realized, two cases need to be considered.

Case 1. First suppose $\sigma_v(r) = \sigma_v(r-) + 1$. By induction, there is a $j \in \mathcal{I}$ such that $\sigma_v(r-) = \zeta_v^j(r-)$. Since ζ_v^j jumps too and by (3.5),

$$\begin{aligned}
\sigma_v(r) &\geq \sup_{i \in \mathcal{I}} \zeta_v^i(r) \geq \zeta_v^j(r) \\
&= \max\bigl[\zeta_v^j(r-) + 1\,,\, \max\{\zeta_{v+z}^j(r-) : z \in \mathcal{N}\}\bigr] \\
&\geq \zeta_v^j(r-) + 1 = \sigma_v(r-) + 1 \\
&= \sigma_v(r)\,.
\end{aligned}$$

Thus ζ_v^j jumps to the same height as σ_v.

Case 2. The second possibility is that for some $w \in \mathcal{N}$, $\sigma_v(r) = \sigma_{v+w}(r-)$. Then by the jump rule (3.1)

$$\sigma_{v+w}(r-) \geq \sigma_v(r-) + 1 \text{ and } \sigma_{v+w}(r-) \geq \sigma_{v+z}(r-) \text{ for all } z \in \mathcal{N}.$$

By the definition of the random graph $v + w \in \mathcal{C}$, so by induction there exists a $j \in \mathcal{I}$ such that $\sigma_{v+w}(r-) = \zeta_{v+w}^j(r-)$. By (3.5)

$$\sigma_v(r-) + 1 \geq \zeta_v^j(r-) + 1 \text{ and } \sigma_{v+z}(r-) \geq \zeta_{v+z}^j(r-) \text{ for } z \in \mathcal{N}.$$

Together these equalities and inequalities yield

$$\zeta_{v+w}^j(r-) \geq \max\bigl[\zeta_v^j(r-) + 1\,,\, \max\{\zeta_{v+z}^j(r-) : z \in \mathcal{N}\}\bigr]\,,$$

so the jump rule (3.1) applied to ζ_v^j gives

$$\zeta_v^j(r) = \zeta_{v+w}^j(r-) = \sigma_{v+w}(r-) = \sigma_v(r).$$

In other words, ζ_v^j jumped to the same level as σ_v, and (3.4) continues to hold right after the jump time r.

The summarize: we have shown that (3.4) holds for times $0 \leq t \leq t_0$ for any processes σ, $\{\zeta^i\}$ that satisfy the hypothesis (3.3) at $t = 0$. Now apply the same step again, to the processes $\tilde{\sigma}(t) = \sigma(t_0{+}t)$ and $\{\tilde{\zeta}^i(t) = \zeta^i(t_0{+}t)\}$. These processes satisfy the hypothesis at $t = 0$ by virtue of (3.4) at $t = t_0$. This way the validity of (3.4) is extended to times $0 \leq t \leq 2t_0$. And so on, to arbitrarily large times. $\square$

For $v \in \mathbf{Z}^d$, let $Z^v = (Z_u^v(t) : u \in \mathbf{Z}^d)$ denote the ballistic deposition process started from a seed in cell $(v, 0)$. In other words, initially

$$Z_u^v(0) = \begin{cases} 0 & \text{if } u = v \\ -\infty & \text{if } u \neq v\,. \end{cases} \tag{3.6}$$

Given an arbitrary initial configuration $\sigma(0) = (\sigma_u(0) : u \in \mathbf{Z}^d)$ [random or deterministic], define a family of processes, indexed by $\mathcal{I} = \mathbf{Z}^d$, by the initial conditions

$$\zeta_u^v(0) = \begin{cases} \sigma_v(0) & \text{if } u = v \\ -\infty & \text{if } u \neq v\,. \end{cases} \tag{3.7}$$

We can write

$$\zeta_u^v(t) = \sigma_v(0) + Z_u^v(t) \tag{3.8}$$

with the convention that $\infty + (-\infty) = -\infty$. The processes σ and $\{\zeta^v : v \in \mathbf{Z}^d\}$ satisfy (3.3). Lemma 3.3 gives this corollary, which is basic for our proof of the hydrodynamic limit:

Corollary 3.1. *The equality*

$$\sigma_u(t) = \sup_{v \in \mathbf{Z}^d} \{\sigma_v(0) + Z_u^v(t)\} \tag{3.9}$$

holds almost surely, for all $u \in \mathbf{Z}^d$ and $t \geq 0$.

4. First-passage site percolation

In this section we prove some estimates for the first-passage percolation problem briefly encountered in the introduction. First we redefine it in the standard way.

Give each site $u \in \mathbf{Z}^d$ an Exp(1)-distributed random time $t(u)$, independently of the other sites. Say $\pi = \{w^0, w^1, \ldots, w^m\} \subseteq \mathbf{Z}^d$ is a *nearest-neighbor path* from u to v of length m if $m < \infty$, $w^0 = u$, $w^m = v$, and $|w^i - w^{i-1}| = 1$ for $i = 1, \ldots, m$. We use $|\cdot|$ to denote the ℓ^1 norm: $|w| = |w_1| + \cdots + |w_d|$ for $w = (w_1, w_2, \ldots, w_d) \in \mathbf{Z}^d$. The passage time of a path $\pi = \{w^0, w^1, \ldots, w^m\}$ is

$$T(\pi) = \sum_{i=1}^{m} t(w^i)\,. \tag{4.1}$$

Since the path starts from site w^0, the value $t(w^0)$ is not included in the sum. The passage time from site u to v is

$$T(u, v) = \inf_{\pi} T(\pi) \tag{4.2}$$

where the infimum ranges over nearest-neighbor paths π from u to v. The minimization has the effect that π may be assumed *self-avoiding* in the sense that there are no repetitions among $\{w^0, w^1, \ldots, w^m\}$.

The cluster growing from a seed at the origin is defined by

$$\mathcal{B}(t) = \{u \in \mathbf{Z}^d : T(0,u) \le t\}\,. \tag{4.3}$$

It is clear from the description that $\mathcal{B}(0) = \{0\}$, and $\mathcal{B}(\cdot)$ grows according to this local rule: Each site adjacent to the current cluster joins independently with rate 1.

To make the connection with ballistic deposition, consider again the process Z started from a seed at the origin:

$$Z_u(0) = \begin{cases} 0, & u = 0 \\ -\infty, & u \neq 0\,. \end{cases} \tag{4.4}$$

Let $R(u,h)$ denote the first time Z is at or above height $h \in \mathbf{Z}_+$ over site $u \in \mathbf{Z}^d$:

$$R(u,h) = \inf\{t > 0 : Z_u(t) \ge h\}. \tag{4.5}$$

In particular, $R(u,0)$ is the first time that a particle sticks to the cluster above site u. $R(0,0) = 0$ by definition (4.4). Since the notation may lead to confusion, let us emphasize that the 0 of $R(u,0)$ is the zero of $\mathbf{Z}_+$, while the 0 of $T(0,u)$ is the origin of $\mathbf{Z}^d$.

Lemma 4.1. *We have the following equalities in distribution between the cluster and passage time processes:*

$$\Big\{\{u \in \mathbf{Z}^d : Z_u(t) \ge 0\} \,:\, t \ge 0\Big\} \overset{d}{=} \big\{\mathcal{B}(t) : t \ge 0\big\} \tag{4.6}$$

and

$$\{R(u,0) : u \in \mathbf{Z}^d\} \overset{d}{=} \{T(0,u) : u \in \mathbf{Z}^d\}\,. \tag{4.7}$$

Proof. Abbreviate $\widetilde{\mathcal{B}}(t) = \{u \in \mathbf{Z}^d : Z_u(t) \ge 0\}$ for this proof. $\mathcal{B}(\cdot)$ and $\widetilde{\mathcal{B}}(\cdot)$ are both Markov jump processes on the countable state space of finite, connected subsets of $\mathbf{Z}^d$ that contain 0. They have the same initial state $\mathcal{B}(0) = \widetilde{\mathcal{B}}(0) = \{0\}$, and both processes have identical infinitesimal rates: each site adjacent to the current cluster joins independently at rate 1. Hence the two processes are equal in distribution, which is (4.6).

By definition (4.3), $T(0,u)$ is the first time when site u joins the cluster $\mathcal{B}(t)$. $R(u,0)$ has the same meaning for $\widetilde{\mathcal{B}}(t)$. So the processes $\{T(0,u)\}$ and $\{R(u,0\}$ are obtained by applying a certain measurable function to the processes $\mathcal{B}(\cdot)$ and $\widetilde{\mathcal{B}}(\cdot)$. Thus (4.7) follows from (4.6). $\square$

In later sections (4.6) will be used several times to derive deviation bounds for Z.

Return to $T(u,v)$ as constructed by (4.3). Subadditivity considerations give a limit

$$\lim_{n\to\infty} \frac{1}{n} T(0,[nx]) = \mu(x) \qquad \text{a.s.} \tag{4.8}$$

for all $x \in \mathbf{R}^d$. The function μ is convex, homogeneous $[\mu(rx) = r\mu(x)$ for $r > 0]$, and Lipschitz continuous. The limiting cluster is defined by

$$B_0 = \{x \in \mathbf{R}^d : \mu(x) \le 1\}\,, \tag{4.9}$$

and now the set convergence (2.3) is valid. We shall not give proofs of these laws of large numbers. The case where the random times $t(u)$ are on the edges instead of on the sites is thoroughly discussed in Kesten's lectures [13]. We prove some large deviation estimates for $T(0,u)$ that we need in the sequel.

Proposition 4.1. *For any $x \in \mathbf{R}^d$ and $\varepsilon > 0$ there are finite constants $C_i = C_i(x,\varepsilon) > 0$ such that*

$$P\big(\big|T(0,[nx]) - n\mu(x)\big| \ge n\varepsilon\big) \le C_1 \exp(-C_2 n)$$

for all n.

Before the proof we derive two corollaries, assuming Proposition 4.1.

Corollary 4.1. *For any $t, \varepsilon > 0$ there are finite constants $C_i = C_i(t,\varepsilon) > 0$ such that*

$$\sum_{u\in\mathbf{Z}^d\,,\,u\notin n(t+\varepsilon)B_0} P\big(T(0,u) \le nt\big) \le C_1 \exp(-C_2 n)$$

for all n.

Proof of Corollary 4.1. Fix a large n and small $\delta_0, \delta_1, \varepsilon_1 > 0$ so that

$$\delta_0 + d/n \le \delta_1 \text{ and } \delta_1(1+\varepsilon_1) \le \varepsilon - \varepsilon_1. \tag{4.10}$$

Let $\mathcal{A}_m = \big[(m+1)(t+\varepsilon)B_0 \setminus m(t+\varepsilon)B_0\big] \cap \mathbf{Z}^d$ for $m \ge n$. Write

$$\sum_{u\notin n(t+\varepsilon)B_0} P\big(T(0,u) \le nt\big) = \sum_{m=n}^{\infty} \sum_{u\in\mathcal{A}_m} P\big(T(0,u) \le nt\big)\,. \tag{4.11}$$

Since $|\mathcal{A}_m| \le Cm^d$, it suffices to bound $P\big(T(0,u) \le nt\big) \le \exp(-Cm)$ for $u \in \mathcal{A}_m$.

Pick and fix points $x^1, \ldots, x^k \in 2(t+\varepsilon)B_0 \setminus (t+\varepsilon)B_0$ such that each $y \in 2(t+\varepsilon)B_0 \setminus (t+\varepsilon)B_0$ is within ℓ^1 distance δ_0 of one of the x^i's. By the definition (4.9), $\mu(x^i) \geq t+\varepsilon$. Any $u \in \mathcal{A}_m$ satisfies

$$|u - [mx^i]| \leq m\delta_0 + d \tag{4.12}$$

for some $1 \leq i \leq k$. For this same i,

$$T(0,[mx^i]) \leq T(0,u) + T(u,[mx^i]) \,.$$

$T(u,[mx^i])$ is stochastically dominated by a sum $S^1_{[m\delta_1]}$ of $[m\delta_1]$ i.i.d. Exp(1) random variables, because by (4.12) a direct lattice path from u to $[mx^i]$ takes at most $m\delta_0 + d \leq m(\delta_0 + d/n) \leq m\delta_1$ Exp(1) passage times. Consequently, by (4.10), Proposition 4.1, and standard large deviation bounds for exponential random variables, for $u \in \mathcal{A}_m$,

$$\begin{aligned}
&P\bigl(T(0,u) \leq nt\bigr) \\
&\leq \sum_{i=1}^{k} P\bigl(T(0,[mx^i]) \leq m\mu(x^i) - m\varepsilon_1\bigr) + kP\bigl(S^1_{[m\delta_1]} \geq m\delta_1(1+\varepsilon_1)\bigr) \\
&\leq C_1 k \exp(-C_2 m)
\end{aligned}$$

for some finite constants $C_1, C_2 > 0$ [not the same as those in Proposition 4.1]. By substituting this bound in (4.11) we have proved Corollary 4.1 for large enough n. By increasing C_1 in the statement it follows for all n. $\square$

Corollary 4.2. *For any $t, \varepsilon > 0$ there are finite constants $C_i = C_i(t,\varepsilon) > 0$ such that*

$$P\bigl(n(t-\varepsilon)B_0 \not\subseteq \mathcal{B}(nt)\bigr) \leq C_1 \exp(-C_2 n)$$

for all n.

Proof. The event $n(t-\varepsilon)B_0 \not\subseteq \mathcal{B}(nt)$ implies that $T(0,u) > nt$ for some $u \in n(t-\varepsilon)B_0$. The idea for proving this is the same as for the previous Corollary: Pick a fine grid of point $x^1, \ldots, x^k \in (t-\varepsilon)B_0$. For each $u \in \bigl[n(t-\varepsilon)B_0\bigr] \cap \mathbf{Z}^d$ pick the closest $[nx^i]$, use the inequality

$$T(0,u) \leq T(0,[nx^i]) + T([nx^i],u) \,,$$

dominate $T([nx^i],u)$ by a sum of i.i.d. Exp(1)'s, note that $\mu(x^i) \leq t-\varepsilon$, and apply Proposition 4.1. We leave the details to the reader. $\square$

As the last item of this section we prove Proposition 4.1. For this purpose we combine a result of Talagrand [30], stated as the next lemma, with some ideas from

Kesten [14]. Suppose $\{X_i : 1 \le i \le N\}$ are independent random variables such that $0 \le X_i \le 1$. Let $\mathcal{F}$ be a family of N-tuples $\alpha = (\alpha_i : 1 \le i \le N)$ of real numbers. Set $\sigma = \sup_{\alpha \in \mathcal{F}} \|\alpha\|_2$, where $\|\alpha\|_2 = \Big(\sum_{i=1}^N \alpha_i^2\Big)^{1/2}$. Define the random variable

$$Z = \sup_{\alpha \in \mathcal{F}} \sum_{i=1}^N \alpha_i X_i \,. \tag{4.13}$$

Lemma 4.2. [30] *For any numbers $a < b$,*

$$P\big(Z \ge b\big) \cdot P\big(Z \le a\big) \le \exp\big[-(b-a)^2/4\sigma^2\big] \,. \tag{4.14}$$

Some comments: We use here the original notation of [30] even though the use of Z, σ and $\mathcal{F}$ conflicts with our notation elsewhere. But this notation is used only for the proof of Proposition 4.1 so it will not cause confusion. Except for Lemma 4.2 our proof of Proposition 4.1 is self-contained, and the proof of Lemma 4.2 can be quickly read from Talagrand's paper [30]: the reader only needs to go through section 4.1 which is self-contained, and the proof of Theorem 8.1.1 which relies on section 4.1. Our Lemma 4.2 is contained in the proof of Theorem 8.1.1 in [30].

Fix $x \in \mathbf{R}^d$. To produce the random variable Z of (4.13), we truncate the random times of the sites and restrict the set of paths π. Let $K, L > 0$ be constants independent of n, to be chosen below. Let Π_L denote the collection of self-avoiding nearest-neighbor paths π from 0 to $[nx]$ with length at most Ln: $\pi = \{0 = w^0, w^1, \dots, w^{m-1}, w^m = [nx]\}$ such that $m \le Ln$. Define truncated times by $\widehat{t}(u) = t(u) \wedge K$ for $u \in \mathbf{Z}^d$. Let $\widehat{T}$ denote the passage time with truncated variables and restricted paths:

$$\widehat{T}(0, [nx]) = \min_{\pi \in \Pi_L} \widehat{T}(\pi) \,,$$

where $\widehat{T}(\pi)$ is defined by (4.1) with t replaced by $\widehat{t}$.

Let $\{0 = u^0, u^1, u^2, \dots, u^N\}$ be a numbering of the sites that are within ℓ^1 distance Ln of the origin. Let $\mathcal{F}$ be the subset of $\{-1, 0\}$-valued N-tuples $\alpha = (\alpha_i)$ that are negatives of indicator functions of paths $\pi \in \Pi_L$: $\alpha \in \mathcal{F}$ iff for some $\pi \in \Pi_L$, $\alpha_i = -\mathbf{1}\{u^i \in \pi\}$ for all $1 \le i \le N$. Define i.i.d. random variables with values in $[0, 1]$ by $X_i = K^{-1}\widehat{t}(u^i)$. The random variable Z defined in (4.13) is then

$$Z = \sup_{\alpha \in \mathcal{F}} \sum_{i=1}^N \alpha_i \frac{\widehat{t}(u^i)}{K} = -\frac{1}{K} \inf_{\pi \in \Pi_L} \sum_{i : u^i \in \pi} \widehat{t}(u^i) = -\frac{1}{K} \widehat{T}(0, [nx]) \,. \tag{4.15}$$

Lemma 4.3. *Given $\varepsilon > 0$, we can choose the constants $L = L(x)$ and $K = K(\varepsilon, x)$ so that*

$$P\big(\big|\widehat{T}(0,[nx]) - T(0,[nx])\big| \geq n\varepsilon \big) \leq C_1 \exp(-C_2 n) \tag{4.16}$$

for all n, for some finite constants $C_1, C_2 > 0$.

Proof. $T(0,[nx])$ has a.s. a unique optimal path π_n because the site times $t(u)$ have a continuous distribution. $\widehat{T}(0,[nx])$ does not necessarily have a unique optimal path, so order the paths in Π_L in some way and let $\widehat{\pi}_n$ denote the first path in this ordering that is optimal for $\widehat{T}(0,[nx])$. Divide the estimation in (4.16) into two parts:

$$\begin{aligned} &P\big(\big|\widehat{T}(0,[nx]) - T(0,[nx])\big| \geq n\varepsilon \big) \\ &\leq P\big(\pi_n \notin \Pi_L\big) + P\big(\big|\widehat{T}(0,[nx]) - T(0,[nx])\big| \geq n\varepsilon\,,\ \pi_n \in \Pi_L \big)\,. \end{aligned}$$

The first term of the right-hand side is bounded as follows:

$$\begin{aligned} &P\big(\pi_n \notin \Pi_L\big) \\ &\leq P\big(T(0,[nx]) \geq nL_0 \big) + P\Bigg(\text{a self-avoiding path } \pi \text{ starts from } 0 \\ &\qquad\qquad \text{and has length at least } Ln \text{ but } T(\pi) < nL_0 \Bigg) \\ &\leq P\big(S^1_{|[nx]|} \geq nL_0 \big) + \sum_{m=Ln}^{\infty} (2d-1)^m P\bigg(S^1_m < m \cdot \frac{L_0}{L} \bigg)\,. \end{aligned}$$

Above we used $T(0,[nx]) \leq S^1_{|[nx]|}$, bounded the number of self-avoiding paths from 0 with length m by $(2d-1)^m$ [at each step at most $2d-1$ sites to choose from], and noted that for a path π of length m, $T(\pi)$ is distributed like S^1_m, a sum of m i.i.d. Exp(1) variables. By choosing $L_0 = x+1$ and $L = L(x) = 2x+2$ the quantity above is bounded by

$$\exp(-C_1 n) + \sum_{m=Ln}^{\infty} (2d-1)^m \exp(-C_2 m) \leq C_1 \exp(-C_2 n)$$

for constants $C_1, C_2 > 0$. Note that the constants change from one inequality to the next.

To bound the probability $P\big(\big|\widehat{T}(0,[nx]) - T(0,[nx])\big| \geq n\varepsilon\,,\ \pi_n \in \Pi_L \big)$ note first that if $\pi_n \in \Pi_L$, then necessarily

$$\begin{aligned} 0 \leq T(0,[nx]) - \widehat{T}(0,[nx]) &\leq \sum_{w \in \widehat{\pi}_n} \big[t(w) - \widehat{t}(w) \big] \\ &= \sum_{w \in \widehat{\pi}_n} [t(w) - K] \cdot \mathbf{1}\{t(w) > K\}\,. \end{aligned}$$

For paths $\pi \in \Pi_L$ define the events

$$A_\pi = \left\{ \sum_{w \in \pi} [t(w) - K] \cdot \mathbf{1}\{t(w) > K\} \geq n\varepsilon \right\} \quad \text{and} \qquad D_\pi = \{\widehat{\pi}_n = \pi\}\,.$$

Let $\mathcal{G}_\pi$ denote the σ-algebra generated by $\{t(w) : w \notin \pi\}$. When π and $\{t(w) : w \notin \pi\}$ are fixed, A_π is an increasing event and D_π is a decreasing event of the variables $\{t(w) : w \in \pi\}$. We estimate as follows:

$$\begin{aligned}
&P\big(\big|\widehat{T}(0,[nx]) - T(0,[nx])\big| \geq n\varepsilon\,,\, \pi_n \in \Pi_L \big) \\
&\leq P\Bigg(\sum_{w \in \widehat{\pi}_n} [t(w) - K] \cdot \mathbf{1}\{t(w) > K\} \geq n\varepsilon \Bigg) \\
&= \sum_{\pi \in \Pi_L} P\big(A_\pi \cap D_\pi\big) \\
&= \sum_{\pi \in \Pi_L} E\big[P\big(A_\pi \cap D_\pi \;\big|\; \mathcal{G}_\pi\big)\big] \\
&\leq \sum_{\pi \in \Pi_L} E\big[P(A_\pi|\mathcal{G}_\pi) P(D_\pi|\mathcal{G}_\pi)\big] \\
&\qquad\qquad\qquad\qquad [\text{ by the FKG inequality }] \\
&= \sum_{\pi \in \Pi_L} P(A_\pi)\, E\big[P(D_\pi|\mathcal{G}_\pi)\big] \\
&\qquad\qquad\qquad\qquad [\ A_\pi \text{ is independent of } \mathcal{G}_\pi\ ] \\
&\leq P\Bigg(\sum_{i=1}^{Ln} [t(u^i) - K] \cdot \mathbf{1}\{t(u^i) > K\} \geq n\varepsilon \Bigg) \cdot \sum_{\pi \in \Pi_L} P(D_\pi) \\
&\qquad\qquad\qquad\qquad [\text{ each } \pi \text{ has at most } Ln \text{ passage times }] \\
&\leq P\Bigg(\sum_{i=1}^{Ln} [t(u^i) - K] \cdot \mathbf{1}\{t(u^i) > K\} \geq n\varepsilon \Bigg)\,.
\end{aligned}$$

This last probability is $\leq \exp(-Cn)$ if we choose $K = K(\varepsilon, L)$ large enough so that $E[(t(u^i) - K) \cdot \mathbf{1}\{t(u^i) > K\}] < \varepsilon/(2L)$. $\square$

We are ready to prove Proposition 4.1. For convenience, replace ε by 8ε in the statement of the Proposition. Fix $\varepsilon > 0$, and choose K, L so that (4.16) holds.

$$\begin{aligned}
&P\big(\big|T(0,[nx]) - n\mu(x)\big| \geq 8n\varepsilon \big) \\
&\leq P\big(\big|T(0,[nx]) - \widehat{T}(0,[nx])\big| \geq 4n\varepsilon \big) \\
&\qquad + P\big(\big|\widehat{T}(0,[nx]) - n\mu(x)\big| \geq 4n\varepsilon \big)\,,
\end{aligned}$$

so by Lemma 4.3 it suffices to bound the last probability. By the limit in (4.8) and by Lemma 4.3,

$$P\big(\,\big|\widehat{T}(0,[nx]) - n\mu(x)\big| \ge 2n\varepsilon\big) < 1/2 \tag{4.17}$$

for all large enough n. Recall equality (4.15), use (4.17) and estimate as follows:

$$\begin{aligned}
&P\big(\,\big|\widehat{T}(0,[nx]) - n\mu(x)\big| \ge 4n\varepsilon\big)\\
&= P\big(\,\widehat{T}(0,[nx]) \ge n\mu(x) + 4n\varepsilon\big)\\
&\qquad + \; P\big(\,\widehat{T}(0,[nx]) \le n\mu(x) - 4n\varepsilon\big)\\
&= P\big(\,Z \le -n\mu(x)/K - 4n\varepsilon/K\big)\\
&\qquad + \; P\big(\,Z \ge -n\mu(x)/K + 4n\varepsilon/K\big)\\
&\le 2\cdot P\big(\,Z \le -n\mu(x)/K - 4n\varepsilon/K\big)\cdot P\big(\,Z \ge -n\mu(x)/K - 2n\varepsilon/K\big)\\
&\qquad + \; 2\cdot P\big(\,Z \ge -n\mu(x)/K + 4n\varepsilon/K\big)\cdot P\big(\,Z \le -n\mu(x)/K + 2n\varepsilon/K\big)\\
&\le 4\exp\!\big[-\big(2n\varepsilon/K\big)^2/4Ln\big]\\
&= 4\exp\!\big[-\varepsilon^2 n/(K^2L)\big]\,.
\end{aligned}$$

We used Lemma 4.2 in the second last step. Each $\alpha \in \mathcal{F}$ has at most Ln entries equal to -1 and the rest are zeroes. Consequently $\sigma \le \sqrt{Ln}\,$. We have proved Proposition 4.1 for large enough n, and it follows for all n by increasing C_1 sufficiently.

5. The ballistic deposition shape from a seed

In this section we prove Theorem 1, the almost sure limit for the ballistic deposition process started from a single seed. Initially

$$Z_u(0) = \begin{cases} 0 & \text{if } u = 0\\ -\infty & \text{if } u \ne 0\,. \end{cases} \tag{5.1}$$

Caution to the reader: Z was defined by (4.15) only for the proof of Proposition 4.1. Elsewhere in the paper, Z denotes the ballistic deposition process started from a seed. The process Z is constructed by the argument of Section 3 on the probability space $(\Omega, \mathcal{F}, P)$ of the Poisson jump time processes $\mathcal{T} = \{\mathcal{T}^u\}$. Recall that $B_0 \subseteq \mathbf{R}^d$ denotes the closed, convex limiting set for first-passage site percolation on $\mathbf{Z}^d$ with $\mathrm{Exp}(1)$ waiting times. The goal is to prove that for a bounded, positive, concave function g defined on the open set $\operatorname{int} B_0$,

$$\lim_{n\to\infty} \frac{1}{n} Z_{[nx]}(nt) = t g(x/t) \qquad \text{a.s.} \tag{5.2}$$

for all $x \in \mathbf{R}^d$ and $t > 0$ such that $x/t \in \operatorname{int} B_0$.

The main tool in the proof is the Kesten-Hammersley lemma from subadditive ergodic theory. Since Z is unbounded both above and below, we work instead with the passage times $R(u, h)$ defined by (4.5). By definition $R(u, h) \geq 0$. Since Z can always reach cell (u, h) by advancing along each coordinate axis in turn, with Exp(1) waiting times,

$$R(u,h) \preccurlyeq S^1_{|u|+h} \,. \tag{5.3}$$

Here $\preccurlyeq$ denotes stochastic dominance and S^1_n is a sum of n i.i.d. Exp(1) random variables. Recall that $|u|$ is the ℓ^1 norm on $\mathbf{Z}^d$. In particular, $R(u, h)$ has finite moments of all order.

The main property to check is a subadditivity:

Lemma 5.1. *For $u, v \in \mathbf{Z}^d$ and $h, k \in \mathbf{Z}_+$, there exists a random variable $\widetilde{R}(v, k)$ such that*

$$R(u+v, h+k) \leq R(u,h) + \widetilde{R}(v,k) \,, \tag{5.4}$$

and $\widetilde{R}(v, k)$ is independent of $R(u, h)$, and equal in distribution to $R(v, k)$.

Proof. To define $\widetilde{R}(v, k)$, we start a new Z-process at time $R(u, h)$ from a seed in cell (u, h). This new process $\widetilde{Z}$ is dominated by the original Z-process, hence the inequality (5.4).

For the reader not familiar with these types of arguments, here are the details: Think of $\mathcal{T} = \{\mathcal{T}^w : w \in \mathbf{Z}^d\}$ as an infinite-dimensional vector of Poisson point processes $\mathcal{T}^w$, indexed by time $t \in (0, \infty)$. Let $\mathcal{F}_t$ be the σ-algebra generated by the restriction of $\mathcal{T}$ to the time-interval $(0, t]$. Then $R(u, h)$ is a stopping time for the filtration $(\mathcal{F}_t)$. Restart the Poisson processes at time $R(u, h)$ and translate the index by u to get new point processes $\widetilde{\mathcal{T}} = \{\widetilde{\mathcal{T}}^w : w \in \mathbf{Z}^d\}$ where $\widetilde{\mathcal{T}}^w = \big[\mathcal{T}^{u+w} - R(u,h)\big] \cap (0, \infty)$. (The subtraction means that epochs of $\mathcal{T}^{u+w}$ are translated back $R(u, h)$ time units.)

Sublemma. *$\widetilde{\mathcal{T}}$ is a collection of i.i.d. rate 1 Poisson processes on $(0, \infty)$, independent of $R(u, h)$.*

Proof of sublemma. We ignore the spatial translation $w \mapsto u + w$. $R = R(u, h)$ is the stopping time, and $\mathcal{F}_R$ is the σ-algebra up to time R, consisting of events A such that $A \cap \{R \leq t\} \in \mathcal{F}_t$ for all t. Topologize the space of families $\mathcal{T} = \{\mathcal{T}^w : w \in \mathbf{Z}^d\}$ of point processes on $(0, \infty)$ by the product of the vague topology on point processes. Let g be a bounded continuous function on that space. The claim is that for any $A \in \mathcal{F}_R$ and any such g,

$$E[\mathbf{1}_A \, g(\widetilde{\mathcal{T}})] = P(A) E[g(\mathcal{T})] \,. \tag{5.5}$$

Since R itself is $\mathcal{F}_R$-measurable, the lemma follows.

Take a sequence of discrete stopping times R_n that decrease down to R, almost surely. For example, $R_n = \sum_k k2^{-n}\mathbf{1}\{(k-1)2^{-n} < R \le k2^{-n}\}$. Let $\widetilde{\mathcal{T}}_n$ denote $\mathcal{T}$ translated back R_n time units and restricted to $(0,\infty)$. $A \in \mathcal{F}_R$ implies $A \in \mathcal{F}_{R_n}$, and we get:

$$\begin{aligned}
E[\mathbf{1}_A\, g(\widetilde{\mathcal{T}}_n)] &= \sum_{i=1}^{\infty} E[\mathbf{1}_A\, \mathbf{1}\{R_n = t_i\} g(\widetilde{\mathcal{T}}_n)] \\
&= \sum_{i=1}^{\infty} E\big[\mathbf{1}_A \mathbf{1}\{R_n = t_i\} g\big((\mathcal{T} - t_i) \cap (0,\infty)\big)\big] \\
&= \sum_{i=1}^{\infty} E[\mathbf{1}_A \mathbf{1}\{R_n = t_i\}] E[g(\mathcal{T})] \\
&= P(A)E[g(\mathcal{T})]\,.
\end{aligned}$$

The $\{t_i\}$ are the possible values of R_n. The point of the above calculation is of course that, when t_i is a deterministic number, $A \cap \{R_n = t_i\} \in \mathcal{F}_{t_i}$, while $\mathcal{T} - t_i$ restricted to $(0,\infty)$ is an i.i.d. collection of Poisson point processes independent of $\mathcal{F}_{t_i}$. Now let $n \to \infty$, and observe that $\widetilde{\mathcal{T}}_n \to \widetilde{\mathcal{T}}$ a.s. so that $E[\mathbf{1}_A\, g(\widetilde{\mathcal{T}}_n)]$ converges to $E[\mathbf{1}_A\, g(\widetilde{\mathcal{T}})]$, and we obtain (5.5) in the limit. $\square$

Return to the proof of Lemma 5.1. Define the process $\widetilde{Z}$ as a function of $\widetilde{\mathcal{T}}$, exactly as Z is a function of $\mathcal{T}$, with an initial seed at the origin: $\widetilde{Z}_w(0) = -\infty \cdot \mathbf{1}\{w \ne 0\}$ as in (5.1) for Z. Let $\widetilde{R}(v,k)$ be the time when $\widetilde{Z}$ gets at or above cell (v,k). Then $\widetilde{R}(v,k)$ is independent of $R(u,h)$, and has exactly the same distribution as $R(v,k)$ defined by (4.5).

Consider the processes σ and ρ defined by

$$\sigma_w(t) = Z_w(R(u,h)+t) \text{ and } \rho_w(t) = h + \widetilde{Z}_{w-u}(t).$$

Then $\rho_w(0) = -\infty$ except at $w = u$, where

$$\rho_u(0) = h + \widetilde{Z}_0(0) = h = Z_u(R(u,h)) = \sigma_u(0).$$

The jump times of σ_w are given by $\mathcal{T}^w - R(u,h)$, and those of ρ_w by $\widetilde{\mathcal{T}}^{w-u} = \mathcal{T}^w - R(u,h)$. So processes σ and ρ use the same Poisson jump times and initially $\sigma(0) \ge \rho(0)$. By Lemma 3.2 $\sigma(t) \ge \rho(t)$ for all t. Take $t = \widetilde{R}(v,k)$. Then

$$\begin{aligned}
Z_{u+v}(R(u,h) + \widetilde{R}(v,k)) &= \sigma_{u+v}(\widetilde{R}(v,k)) \\
&\ge \rho_{u+v}(\widetilde{R}(v,k)) = h + \widetilde{Z}_v(\widetilde{R}(v,k)) \\
&= h + k\,,
\end{aligned}$$

which says that Z_{u+v} has reached height $h+k$ by time $R(u,h)+\widetilde{R}(v,k)$. This implies (5.4) and completes the proof of Lemma 5.1. $\square$

Inequality (5.4), the existence of moments, and the Kesten-Hammersley lemma as given on p. 20 of [28] imply that for all $u \in \mathbf{Z}^d$ and $h \in \mathbf{Z}_+$ there exists a number $\gamma(u,h)$ such that, for any positive integer m,

$$\lim_{j\to\infty} \frac{1}{2^j m} R(2^j mu, 2^j mh) = \gamma(u,h) \qquad \text{a.s.} \tag{5.6}$$

The first task is to extend (5.6) to a genuine limit, and for that we use the following continuity property of the passage times. It is an immediate consequence of (5.3), (5.4) and the monotonicity of $R(u,h)$ in the h-variable.

Lemma 5.2. *For any $u, v \in \mathbf{Z}^d$ and $h, k \in \mathbf{Z}_+$,*

$$R(u,h) - R(v,k) \preccurlyeq S^1_{|u-v|+(h-k)^+} \,. \tag{5.7}$$

Now regard (u,h) fixed, and also fix $\varepsilon > 0$ and an integer m large enough relative to ε and (u,h) [how large m needs to be is seen shortly]. For large enough n, pick first $j = j(n)$ so that $2^j m \le n < 2^{j+1} m$. Then it is possible to find a $k = k(n) \in \{0, 1, \ldots, m-1\}$ such that

$$2^j(m+k) \le n < 2^j(m+k+1). \tag{5.8}$$

We write

$$\begin{aligned} &\frac{2^j(m+k)}{n} \cdot \frac{1}{2^j(m+k)} R\big(2^j(m+k)u, 2^j(m+k)h\big) \\ &\le \frac{1}{n} R(nu,nh) + \frac{1}{n} U(n)\,, \end{aligned} \tag{5.9}$$

where the error

$$U(n) \equiv \big\{ R\big(2^j(m+k)u, 2^j(m+k)h\big) - R(nu,nh) \big\} \vee 0 \preccurlyeq S^1_{2^j L}$$

for a constant $L = L(u,h)$, by (5.7). The Cramér rate function for Exp(1) is $\kappa(x) = x - 1 - \log x$, so we get the estimate

$$\begin{aligned} P\big(U(n) \ge n\varepsilon\big) &\le P\left(S^1_{2^j L} \ge 2^j L \frac{n\varepsilon}{2^j L} \right) \\ &\le P\left(S^1_{2^j L} \ge 2^j L m\varepsilon / L \right) \\ &\le \exp\big[-2^j L\, \kappa\big(m\varepsilon/L\big)\big] \\ &\le \exp[-Cn]\,, \end{aligned} \tag{5.10}$$

where $C > 0$ is a constant. In the above calculation we used $2^j m \le n < 2^{j+1}m$ and $\kappa(x) \ge x/2$ for large enough x, and took m large enough.

Let $n \to \infty$ in (5.9), so that $j \to \infty$ also. Even though $k = k(n)$ varies with n, it has only finitely many possible values so the limit (5.6) happens on the left-hand side of (5.9). Note that

$$\frac{2^j(m+k)}{n} \ge \frac{m+k}{m+k+1} \ge \frac{m}{m+1}\,.$$

The error $n^{-1}U(n)$ vanishes a.s. by the estimate (5.10) and Borel-Cantelli. We get

$$\liminf_{n\to\infty} \frac{1}{n} R(nu, nh) \ge \frac{m}{m+1}\gamma(u,h) \qquad \text{a.s.}$$

A similar argument works for the limsup. Let $m \to \infty$, and we have this intermediate statement: for all $u \in \mathbf{Z}^d$ and $h \in \mathbf{Z}_+$ there exists a number $\gamma(u,h)$ such that

$$\lim_{n\to\infty} \frac{1}{n} R(nu, nh) = \gamma(u,h) \qquad \text{a.s.} \tag{5.11}$$

From (5.7) we get a Lipschitz property for γ:

$$|\gamma(u,h) - \gamma(v,k)| \le |u-v| + |h-k|\,, \tag{5.12}$$

while (5.11) gives homogeneity: for nonnegative integers m

$$\gamma(mu, mh) = m\gamma(u,h)\,. \tag{5.13}$$

(5.13) permits us to define γ unambiguously for $(x,b) \in \mathbf{Q}^d \times \mathbf{Q}_+$ by

$$\gamma(x,b) = \frac{1}{m}\gamma(mx, mb) \tag{5.14}$$

where m is any positive integer such that $(mx, mb) \in \mathbf{Z}^d \times \mathbf{Z}_+$. By an estimation similar to that used in (5.9), the limit in (5.11) can be extended to all $(u,h) \in \mathbf{Q}^d \times \mathbf{Q}_+$.

The final step is to extend the limit in (5.11) so that it holds outside a single exceptional P-null set for all $(u,h) = (x_0, b_0) \in \mathbf{R}^d \times \mathbf{R}_+$. The Lipschitz property (5.12) continues to hold on $\mathbf{Q}^d \times \mathbf{Q}_+$ for the extension of γ defined by (5.13), so we can extend γ uniquely to a Lipschitz function on $\mathbf{R}^d \times \mathbf{R}_+$. For rational (x,b), integers n, and rational $\delta > 0$ define the event $A_{n,\delta}(x,b)$ by

$$\begin{aligned} A_{n,\delta}(x,b) = \Big\{ &\text{there exists } (v,k) \in \mathbf{Z}^d \times \mathbf{Z}_+ \text{ such that} \\ &\big|[nx] - v\big| + \big|[nb] - k\big| \le n\delta \text{ but} \\ &\big|R\big([nx],[nb]\big) - R(v,k)\big| \ge 2n\delta \Big\}\,. \end{aligned} \tag{5.15}$$

By (5.7) and by standard large deviation bounds for exponential random variables,

$$P\big(A_{n,\delta}(x,b)\big) \le C_1 n^{d+1} \exp[-C_2 n]$$

for finite constants $C_i = C_i(x,b,\delta) > 0$. This bound is suitable for Borel-Cantelli. Thus at this stage the following holds with probability 1: for each $(x,b) \in \mathbf{Q}^d \times \mathbf{Q}_+$ and rational $\delta > 0$, (5.11) holds with $(u,h) = (x,b)$, and for large enough n,

$$\big|R\big([nx],[nb]\big) - R(v,k)\big| \le 2n\delta \tag{5.16}$$

for all cells (v,k) that satisfy

$$\big|[nx] - v\big| + \big|[nb] - k\big| \le n\delta. \tag{5.17}$$

Now let $(x_0,b_0) \in \mathbf{R}^d \times \mathbf{R}_+$. Choose rational (x,b) such that $|x-x_0|+|b-b_0| \le \delta/2$. Take $(v,k) = ([nx_0],[nb_0])$. Then (5.17) holds for large enough n, and by letting $n \to \infty$ in (5.16) we get

$$\gamma(x,b) - 2\delta \le \liminf_{n\to\infty} \frac{1}{n} R([nx_0],[nb_0]) \le \limsup_{n\to\infty} \frac{1}{n} R([nx_0],[nb_0]) \le \gamma(x,b) + 2\delta.$$

Take rational (x,b) that converge to (x_0,b_0), and use the Lipschitz continuity of γ. This proves the limit in this proposition:

Proposition 5.1. *There is a homogeneous, subadditive, convex Lipschitz function γ on $\mathbf{R}^d \times \mathbf{R}_+$ such that, outside an event of probability zero,*

$$\lim_{n\to\infty} \frac{1}{n} R([nx],[nb]) = \gamma(x,b) \tag{5.18}$$

for all $(x,b) \in \mathbf{R}^d \times \mathbf{R}_+$.

To prove Proposition 5.1, it remains to argue the properties of γ: The subadditivity (5.4) implies the corresponding subadditivity for γ. This subadditivity is preserved by the extensions of γ to rational and real points of $\mathbf{R}^d \times \mathbf{R}_+$. Same is true of the homogeneity (5.13). Homogeneity and subadditivity together imply convexity.

At this point we have not ruled out the possibility that $\gamma \equiv 0$. (5.3) gives the upper bound

$$\gamma(x,b) \le |x| + b \qquad \text{for all } (x,b) \in \mathbf{R}^d \times \mathbf{R}_+. \tag{5.19}$$

To get a positive lower bound for γ, we construct another growth process in $\mathbf{Z}^d \times \mathbf{Z}_+$ whose height dominates Z, and that has a simple structure so that its spread is easier to bound.

Instead of just focusing on the height $Z_u(t)$ of the growing cluster, let us denote by $\mathcal{Z}(t)$ the actual set of occupied cells at time t. $\mathcal{Z}(t)$ is a subset of $\mathbf{Z}^d \times \mathbf{Z}_+$, and initially $\mathcal{Z}(0) = \{(0,0)\}$. The rule of evolution for $\mathcal{Z}(\cdot)$ is this: At epochs of $\mathcal{T}^u$, the top growth cell above u is annexed to $\mathcal{Z}$. By definition, the top growth cell above site u is $(u,k) \in \mathbf{Z}^d \times \mathbf{Z}_+$ with maximal k subject to the condition that

$$(u,k) \notin \mathcal{Z}, \text{ but either } (u,k-1) \in \mathcal{Z} \text{ or } (u+z,k) \in \mathcal{Z} \text{ for some } z \in \mathcal{N}. \tag{5.20}$$

If no finite k satisfies this condition, nothing is annexed to $\mathcal{Z}$.

Define a different growing cluster $\mathcal{W}$ by stipulating that *all* growth cells (and not just the top one) join independently with rate 1. A cell (u,k) is a growth cell for $\mathcal{W}$ if (5.20) holds with $\mathcal{W}$ instead of $\mathcal{Z}$. So note in particular that the cluster $\mathcal{W}$ grows "sideways" and "up" in $\mathbf{Z}^d \times \mathbf{Z}_+$, but not down. Set initially $\mathcal{W}(0) = \{(0,0)\}$.

To construct $\mathcal{W}$ we employ a collection $\{\mathcal{T}^u_q : u \in \mathbf{Z}^d, q \in \mathbf{Z}_+\}$ of i.i.d. Poisson point processes. The nonnegative integer q labels the growth cells from top down, so that the top growth cell above site u is assigned label $q = 0$, the next highest growth cell gets label $q = 1$, and so on. More formally, we can define

$$g_{u,0} = \max\{k : (u,k) \text{ is a growth cell for } \mathcal{W}\}$$

("g" for growth) and inductively for $q \geq 1$

$$g_{u,q} = \max\{0 \leq k < g_{u,q-1} : (u,k) \text{ is a growth cell for } \mathcal{W}\}\,.$$

By convention, the maximum of an empty set is $-\infty$, so if the current cluster $\mathcal{W}$ has exactly m growth cells above site u, then $g_{u,q} = -\infty$ for $q \geq m$. The $g_{u,q}$'s are of course functions of time too. The precise rule of evolution for $\mathcal{W}$ is this:

$$\begin{array}{l} \text{if } r \text{ is an epoch of } \mathcal{T}^u_q \text{ and } g_{u,q} > -\infty, \text{ then } \mathcal{W}(r) = \mathcal{W}(r-) \cup \{(u, g_{u,q})\}. \\ \text{Update the } g_{v,q}\text{'s for } v = u \text{ and } v \in u + \mathcal{N}. \end{array} \tag{5.21}$$

The argument given for the construction of ballistic deposition in Section 3 does not work for $\mathcal{W}$ because each site u now has infinitely many Poisson processes $\{\mathcal{T}^u_q : q \in \mathbf{Z}_+\}$ attached to it. However, we can easily see that, given any finite time t_1, at most finitely many $\mathcal{T}^u_q$-processes are involved in constructing the dynamics up to time t_1: Starting with the seed at $(0,0)$, let $\tau_1, \tau_2, \tau_3, \ldots$ be the successive waiting times for adding the second, third, fourth,... cell to the existing cluster. Since each new particle adds at most $2d+1$ growth cells, τ_n is stochastically larger than an $\text{Exp}(2dn+n)$ random variable. Consequently $\sum_n \tau_n = \infty$ a.s. and only finitely many steps are needed (and only finitely many Poisson processes $\mathcal{T}^u_q$ inspected) to construct $\mathcal{W}(t)$ for $0 \leq t \leq t_1$.

Now couple $\mathcal{W}(t)$ and $\mathcal{Z}(t)$ by letting $\{\mathcal{T}_0^u\}$ be the Poisson processes that govern the evolution of $\mathcal{Z}$. In other words, both $\mathcal{Z}$ and $\mathcal{W}$ annex their top growth cell above u simultaneously at epochs of $\mathcal{T}_0^u$. At epochs of $\mathcal{T}_q^u$ for $q \geq 1$ the $\mathcal{Z}$-process does nothing, while $\mathcal{W}$ may add other growth cells as stipulated in rule (5.21). Since the top growth cells are not necessarily the same for $\mathcal{Z}$ and $\mathcal{W}$ it does not follow that $\mathcal{W}(t)$ contains $\mathcal{Z}(t)$, but it does follow that the height of $\mathcal{W}$ always dominates the height of $\mathcal{Z}$. If we let

$$W_u(t) = \max\{k \geq 0 : (u,k) \in \mathcal{W}(t)\}$$

when some (u,k) lies in $\mathcal{W}(t)$ and $W_u(t) = -\infty$ otherwise, we get this inequality:

$$Z_u(t) \leq W_u(t) \qquad \text{for all } u \in \mathbf{Z}^d \text{ and } t \geq 0, \text{ a.s.} \tag{5.22}$$

Proof of (5.22) is by induction on jumps [which are only finitely many in any finite time interval, almost surely, as argued above]. As long as (5.22) holds, no top growth cell of $\mathcal{Z}$ can be above the corresponding top growth cell of $\mathcal{W}$, and consequently the next jump preserves (5.22).

To make use of (5.22) we redefine $\mathcal{W}$ as a first-passage problem. Give the cells i.i.d. Exp(1) random waiting times $\{t(u,h) : (u,h) \in \mathbf{Z}^d \times \mathbf{Z}_+\}$. Consider self-avoiding nearest-neighbor lattice paths $\pi = \{(v^0,k^0),(v^1,k^1),\ldots,(v^m,k^m)\}$ whose admissible steps are of these types: for each $i = 1,\ldots,m$,

$$(v^i,k^i) - (v^{i-1},k^{i-1}) = (0,1) \text{ or } (\pm\hat{e}_p,0) \text{ for some } p = 1,\ldots,d, \tag{5.23}$$

where $\hat{e}_1, \ldots, \hat{e}_d$ are the d standard basis vectors in $\mathbf{R}^d$. In other words, inside a layer $\mathbf{Z}^d \times \{k\}$ an admissible path π takes arbitrary nearest-neighbor steps subject to self-avoidance, and across the layers π moves only up, not down. The admissible steps are chosen to match (5.20). The passage time of such a path π is $M(\pi) = \sum_{i=1}^m t(v^i,k^i)$. The passage time of cell (u,h) is

$$M(u,h) = \inf_\pi M(\pi)$$

where the infimum ranges over paths π of the above type from $(0,0) = (v^0,k^0)$ to $(v^m,k^m) = (u,h)$. Let

$$\widetilde{\mathcal{W}}(t) = \{(u,h) : M(u,h) \leq t\}$$

be the growing cluster. For $\widetilde{\mathcal{W}}$ we get a bound easily by counting self-avoiding paths.

Lemma 5.3. *For any finite constant β, there exists a positive $\nu = \nu(\beta)$ such that for any finite integer K,*

$$P\Big(\widetilde{\mathcal{W}}(t) \text{ contains a cell } (u,h) \text{ such that } |u| + h \geq K\Big) \leq 2\exp(-K\beta) \tag{5.24}$$

as long as $t \leq K\nu$.

Proof. Let π denote an admissible path fixed to start at the origin $(0,0) = (v^0, k^0)$. The probability in (5.24) is at most

$$\begin{aligned}
&P\Big(\text{there exists a } \pi \text{ through at least } K \text{ cells with } M(\pi) \leq t\Big) \\
&\leq \sum_{j=K}^{\infty} P\Big(\text{there exists a } \pi \text{ through exactly } j \text{ cells with } M(\pi) \leq t\Big) \\
&\leq \sum_{j=K}^{\infty} (2d+1)^j P\big(S_j^1 \leq t\big) \\
&\leq \sum_{j=K}^{\infty} \exp\Big\{-j\big[\kappa(t/K) - \log(2d+1)\big]\Big\} \\
&\leq \sum_{j=K}^{\infty} \exp(-\beta j) \leq 2\exp(-K\beta)\,.
\end{aligned}$$

We used again the Cramér rate function $\kappa(x) = x - 1 - \log x$ for the Exp(1) distribution. The inequality $P(S_j^1 \leq t) \leq \exp[-j\kappa(t/j)]$ is valid for $t \leq j$. Take $t/K \leq \nu$ with ν small enough so that $\kappa(\nu) \geq \beta + \log(2d+1) + 1$. $\square$

Lemma 5.4. *The processes $\mathcal{W}(\cdot)$ and $\widetilde{\mathcal{W}}(\cdot)$ are equal in distribution.*

Proof. We can regard both processes as jump processes on the countable state space of finite connected subsets of $\mathbf{Z}^d \times \mathbf{Z}_+$ that contain the origin. Both start from $\{(0,0)\}$. Both processes add new admissible cells independently at rate 1. Comparison of (5.20) and (5.23) shows that an admissible new cell, or growth cell, is the same for both $\mathcal{W}(\cdot)$ and $\widetilde{\mathcal{W}}(\cdot)$. Thus the two processes have identical infinitesimal rates. $\square$

Combining (5.22) and Lemmas 5.3 and 5.4, we get a lower bound for γ:

Lemma 5.5. *There exists a positive constant ν such that $\gamma(x,b) \geq \nu(|x| + b)$ for all $(x,b) \in \mathbf{R}^d \times \mathbf{R}_+$.*

Proof. Pick $\nu < \nu(1) =$ the constant given by Lemma 5.3 for $\beta = 1$, and set $t = \nu(|x| + b)$. Then by Lemma 5.3 for $K = |[nx]| + [nb]$,

$$\begin{aligned} P\Big(R\big([nx],[nb]\big) \le nt\Big) &\le P\Big(Z_{[nx]}(nt) \ge [nb]\Big) \\ &\le P\Big(W_{[nx]}(nt) \ge [nb]\Big) \\ &\le 2\exp\big(-|[nx]| - [nb]\big). \end{aligned}$$

Thus $R\big([nx],[nb]\big) \ge nt$ for large enough n, a.s. $\square$

Now we have γ bounded both above and below. Finally, we convert the limit in Proposition 5.1 to that of Theorem 1. We need one more property for γ:

Lemma 5.6. *For $x \in \operatorname{int} B_0$, there is a unique finite $h > 0$ such that $\gamma(x,h) = 1$.*

Proof. For this proof we connect ballistic deposition on $\mathbf{Z}^d \times \mathbf{Z}_+$ with first-passage site percolation on $\mathbf{Z}^d$. Recall the definition of the limit $\mu(x)$ in (4.8). By (4.7) and Proposition 5.1, $\mu(x) = \gamma(x,0)$. By definition, $B_0 = \{x : \mu(x) \le 1\}$, so by homogeneity $\mu(x) < 1$ for $x \in \operatorname{int} B_0$. Hence for such x also $\gamma(x,0) < 1$. By Lemma 5.5 and the continuity of γ there is some $h > 0$ such that $\gamma(x,h) = 1$, so it remains to rule out the possibility of having $0 < h_0 < h_1$ such that $\gamma(x,h_0) = \gamma(x,h_1) = 1$. But this and convexity would imply $\gamma(x,0) \ge 1$, contradicting what was just concluded. $\square$

Proof of Theorem 1. By the previous lemma, a positive function g on $\operatorname{int} B_0$ is uniquely defined by the equation $\gamma(x, g(x)) = 1$. The lower bound of Lemma 5.5 gives an upper bound for g, and g is concave by the convexity of γ. A finite concave function on an open convex set is continuous by Theorem 10.1 in [Rf]. By the homogeneity of γ,

$$\gamma\big(x, tg(x/t)\big) = t. \tag{5.25}$$

By the uniqueness in Lemma 5.6 and the monotonicity of γ, $h > tg(x/t)$ [$h < tg(x/t)$] implies $\gamma(x,h) > t$ [$\gamma(x,h) < t$].

Fix $x \in \mathbf{R}^d$ and $t > 0$ so that $x/t \in \operatorname{int} B_0$. Let $\delta_0 > 0$, and pick $\delta_1 \in (0, \delta_0)$. Set $b = tg(x/t) - \delta_1$, and pick $\varepsilon > 0$ so that $\gamma(x,b) < t - \varepsilon$. Fix a large number m_0 so

that $m_0(\delta_0 - \delta_1) > 1$ to take care of the effects of integer parts.

$$\begin{aligned}
&\left\{\liminf_{n\to\infty} n^{-1} Z_{[nx]}(nt) < tg(x/t) - \delta_0\right\} \\
\subseteq\ &\bigcap_{m=m_0}^{\infty} \bigcup_{n=m}^{\infty} \left\{Z_{[nx]}(nt) < [ntg(x/t) - n\delta_1]\right\} \\
\subseteq\ &\bigcap_{m=m_0}^{\infty} \bigcup_{n=m}^{\infty} \left\{R([nx],[nb]) > n\gamma(x,b) + n\varepsilon\right\} \\
\subseteq\ &\left\{\limsup_{n\to\infty} n^{-1} R([nx],[nb]) > \gamma(x,b) + \varepsilon\right\}.
\end{aligned}$$

The last event has probability zero by Proposition 5.1. Similarly we show that

$$\limsup_{n\to\infty} n^{-1} Z_{[nx]}(nt) \le tg(x/t) \qquad \text{a.s.},$$

and the limit in Theorem 1 is proved. The invariances of g follow from the corresponding invariances in the distribution of Z. $\square$

6. The shape from a translated seed

This section is a purely technical extension of the limit from a seed (Theorem 1) derived in the previous section. The proof of the hydrodynamic limit (Theorem 2) uses the coupling (3.9) which forces us to consider simultaneously the whole family $\{Z^v\}$ of processes. Recall that Z^v stands for the process that grows from a seed in cell $(v, 0)$, as defined by (3.6). We need a limit where the initial seed and the initial time point are translated as the limit is taken.

In Section 3 we defined the processes $\{Z^v\}$ on the probability space $(\Omega, \mathcal{F}, P)$ of the Poisson jump time processes $\mathcal{T} = \{\mathcal{T}^u\}$. Now we extend the Poisson processes to the entire real line $(-\infty, \infty)$. Since Poisson points on $(-\infty, 0]$ and $(0, \infty)$ are independent, this is the same as replacing the original probability space $(\Omega, \mathcal{F}, P)$ with a product space

$$(\overline{\Omega}, \overline{\mathcal{F}}, \overline{P}) = (\Omega^0 \times \Omega, \mathcal{F}^0 \otimes \mathcal{F}, P^0 \otimes P)$$

where $(\Omega^0, \mathcal{F}^0, P^0)$ is the probability space of the Poisson processes on $(-\infty, 0]$. A sample point of the product space is $\overline{\omega} = (\omega^0, \omega)$, where ω still represents a realization of the i.i.d. collection $\mathcal{T} = \{\mathcal{T}^u : u \in \mathbf{Z}^d\}$ of Poisson point processes on $(0, \infty)$, while ω^0 represents a realization of these processes on $(-\infty, 0]$. The

processes Z^v are defined on the space $\overline{\Omega}$ in the obvious way, by ignoring the ω^0-component: $Z_u^v(t)(\omega^0, \omega) = Z_u^v(t)(\omega)$.

Let θ_s for $-\infty < s < \infty$ denote a time translation on the space $\overline{\Omega}$, so that the epochs of $\theta_s \overline{\omega}$ are those of $\overline{\omega}$ translated s time units backward. (In other words, around the time origin $\theta_s \overline{\omega}$ looks like $\overline{\omega}$ around time point s.) The random variable $Z_u^v(t) \circ \theta_s$ is computed by reading Poisson jump times from time point s onwards and by letting the ballistic deposition evolve for a duration t. The passage times are defined as before:

$$R^v(u,h) \circ \theta_s = \inf\{t > 0 : Z_u^v(t) \circ \theta_s \geq h\}\,.$$

Notice that $R^v(u,h) \circ \theta_s$ is the *amount of time* it takes to grow up to height h, and not the time point of the Poisson processes when this happens. Note further that if $s < 0$ then the random variable $Z_u^v(t) \circ \theta_s$ uses the Poisson processes also for negative times.

Theorem 4. *Let g be the limiting function of Theorem 1. Then the following holds with probability 1: For all $x, y \in \mathbf{R}^d$, $s \in \mathbf{R}$, and $t > 0$ such that $y \in x + t \cdot \operatorname{int} B_0$,*

$$\lim_{n\to\infty} \frac{1}{n} Z_{[nx]}^{[ny]}(nt) \circ \theta_{ns} = tg\left(\frac{x-y}{t}\right). \tag{6.1}$$

The remainder of this section proves Theorem 4. We construct step by step an event Γ on $\overline{\Omega}$ that satisfies $\overline{P}(\Gamma) = 1$ and on which the convergence (6.1) holds. As in Section 5, we prove the theorem by proving the convergence of the passage times:

$$\lim_{n\to\infty} \frac{1}{n} R^{[ny]}\big([nx],[nb]\big) \circ \theta_{ns} = \gamma(x-y,b) \tag{6.2}$$

for all $y \in \mathbf{R}^d$, $(x,b) \in \mathbf{R}^d \times \mathbf{R}_+$, and real s.

The reader who is willing to accept the result may ignore the proof without loss of continuity and proceed to the next section where the hydrodynamic limit is proved.

Step 1. First we define Γ as the event on which

$$\lim_{j\to\infty} \frac{1}{2^j m} R^{2^j mv}\big(2^j mu, 2^j mh\big) \circ \theta_{2^j m s_0} = \gamma(u-v,h) \tag{6.3}$$

for all $v \in \mathbf{Z}^d$, $(u,h) \in \mathbf{Z}^d \times \mathbf{Z}_+$, rational s_0, and all $m \in \mathbf{N}$.

We need to argue that $\overline{P}(\Gamma) = 1$. The proof of the Kesten-Hammersley theorem on p. 20–23 of [28] shows that convergence along powers of 2 only depends

on the distributions of the random variables. Let F_n denote the distribution of $R^{nv}(nu, nh) \circ \theta_{ns_0}$. Then F_n is also the distribution of $R(n(u-v), nh)$, and inequality (5.4) shows $F_{m+n} \ge F_m \star F_n$. This, and the existence of second moments, is what is needed for the almost sure convergence in (6.3).

Step 2. Now define Γ to be the event where the requirement of *Step 1* holds, and in addition properties (6.4)–(6.5) below, which are to hold for all $y \in \mathbf{Q}^d$, $(x, b) \in \mathbf{Q}^d \times \mathbf{Q}_+$, rational s_0, and rational $s_1, \delta > 0$:

(6.4) For large enough n, $\bigl|R^{[ny]}\bigl([nx], [nb]\bigr) - R^{[ny]}(v, k)\bigr| \le 2n\delta$ for all cells (v, k) that satisfy $\bigl|[nx] - v\bigr| + \bigl|[nb] - k\bigr| \le n\delta$.

(6.5) For $s_1 > 0$ there exists $\delta_0(s_1) > 0$ such that if $\delta < \delta_0(s_1)$, then for large enough n, $Z_v^{[ny]}(ns_1) \circ \theta_{ns_0} \ge 0$ for all sites v such that $|v - [ny]| \le n\delta$.

Still $\overline{P}(\Gamma) = 1$. We already argued around (5.15)–(5.17) that (6.4) can be satisfied almost surely. By Lemma 4.1, condition (6.5) can be viewed as a percolation question: For each n define a first-passage percolation cluster centered at $[ny]$ by

$$\mathcal{B}(t) = \{v \in \mathbf{Z}^d : Z_v^{[ny]}(t) \circ \theta_{ns_0} \ge 0\}\,.$$

If $\delta_0(s_1)$ is chosen so that $|v - [ny]| \le n\delta_0(s_1)$ implies $v \in [ny] + n(s_1/2)B_0$, then (6.5) holds a.s. by Corollary 4.2 and the Borel-Cantelli lemma.

The next step is to improve the convergence in (6.3) to a genuine limit on the event Γ. Fix m for the moment. As in (5.8), for large enough n there are $j = j(n)$ and $k = k(n) \in \{0, 1, \ldots, m-1\}$ such that

$$q(n) \equiv 2^j(m+k) \le n < 2^j(m+k+1). \tag{6.6}$$

Keeping $v \in \mathbf{Z}^d$, $(u, h) \in \mathbf{Z}^d \times \mathbf{Z}_+$ and $s_0 \in \mathbf{Q}$ fixed, we use properties (6.4) and (6.5) of Γ to write

$$\begin{aligned}
&R^{q(n)v}\bigl(q(n)u, q(n)h\bigr) \circ \theta_{q(n)(s_0 - s)} \\
&\le (n - q(n))s_0 + q(n)s + R^{nv}\bigl(q(n)u, q(n)h\bigr) \circ \theta_{ns_0} \\
&\le (n - q(n))s_0 + ns + R^{nv}\bigl(nu, nh\bigr) \circ \theta_{ns_0} + 2\delta n\,.
\end{aligned} \tag{6.7}$$

This argument will be used several times, so we go over it once carefully. The first inequality in (6.7) is valid on the event Γ for a certain $s = s(m) > 0$ that satisfies $\lim_{m\to\infty} s(m) = 0$, and for all large enough n, by property (6.5) for this reason: The ballistic deposition process $Z^{q(n)v} \circ \theta_{q(n)(s_0 - s)}$ starts from a seed in cell $(q(n)v, 0)$ when the Poisson process clock is at $q(n)(s_0 - s)$. One way for this process to reach cell $\bigl(q(n)u, q(n)h\bigr)$ is to first spend at most time $(n - q(n))s_0 + q(n)s$ to reach cell $(nv, 0)$, and from there follow a new process $Z^{nv} \circ \theta_{ns_0}$ that starts when the Poisson

process clock is at ns_0. Since $0 \le n - q(n) \le n/m$, there is a $\delta = \delta(m) > 0$ such that $\lim_{m\to\infty} \delta(m) = 0$ and $|q(n)v - nv| \le n\delta$ if n is large enough. Hence by (6.5) we can choose $s = s(m) > 0$ such that $Z^{q(n)v} \circ \theta_{q(n)(s-s_0)}$ reaches cell $(nv, 0)$ in time $q(n)s$ if n is large enough.

The second inequality in (6.7) follows from (6.4), again for a $\delta = \delta(m) > 0$ such that $\lim_{m\to\infty} \delta(m) = 0$, and for large enough n. Also $q(n) \le n$ was used. A note about terminology: We say that Z reaches cell (u, h) in time t if $Z_u(t) \ge h$.

Let $n \to \infty$ in (6.7), use the limit (6.3) and that $n - q(n) \le n/m$ to conclude that on Γ,

$$\liminf_{n\to\infty} \frac{1}{n} R^{nv}(nu, nh) \circ \theta_{ns_0} \ge \frac{m-1}{m}\,\gamma(u - v, h) - (s + 2\delta) - \frac{s_0}{m}\,. \tag{6.8}$$

Now let $m \to \infty$, and concurrently we can take $s \to 0$ and $\delta \to 0$.

To handle the limsup we use the same argument in a reverse way:

$$\begin{aligned} &R^{nv}\big(nu, nh\big) \circ \theta_{ns_0} \\ &\le (q(n) - n)s_0 + q(n)s + R^{q(n)v}\big(nu, nh\big) \circ \theta_{q(n)(s+s_0)} \\ &\le ns + R^{q(n)v}\big(q(n)u, q(n)h\big) \circ \theta_{q(n)(s+s_0)} + 2\delta q(n)\,. \end{aligned} \tag{6.9}$$

Now the thinking goes like this: The process $Z^{nv} \circ \theta_{ns_0}$ starts from a seed in cell $(nv, 0)$ when the Poisson process clock is at ns_0. One way for this process to reach cell (nu, nh) is to first spend at most time $(q(n) - n)s_0 + q(n)s$ to reach cell $(q(n)v, 0)$, and from there follow a new process $Z^{q(n)v} \circ \theta_{q(n)(s+s_0)}$ that starts when the Poisson process clock is at $q(n)(s + s_0)$. Again, there is a $\delta = \delta(m) > 0$ such that $\lim_{m\to\infty} \delta(m) = 0$ and $|q(n)v - nv| \le n\delta$ if n is large enough. By (6.5) we can choose $s = s(m) > 0$ such that $\lim_{m\to\infty} s(m) = 0$ and $Z^{nv} \circ \theta_{ns_0}$ reaches cell $(q(n)v, 0)$ in time $(q(n) - n)s_0 + q(n)s$ if n is large enough. The second inequality in (6.9) follows from (6.4).

Let $n \to \infty$ in (6.9) to conclude that on the event Γ,

$$\limsup_{n\to\infty} \frac{1}{n} R^{nv}(nu, nh) \circ \theta_{ns_0} \le \frac{m-1}{m}\,\gamma(u - v, h) + (s + 2\delta)\,. \tag{6.10}$$

Let $m \to \infty$, and concurrently we can take $s \to 0$ and $\delta \to 0$.

The limits (6.8) and (6.10) permit us to strengthen the definition of Γ, without losing $\overline{P}(\Gamma) = 1$:

Step 3. The requirements of *Step 2* hold on Γ, and also

$$\lim_{n\to\infty} \frac{1}{n} R^{nv}\big(nu, nh\big) \circ \theta_{ns_0} = \gamma(u - v, h) \tag{6.11}$$

for all $v \in \mathbf{Z}^d$, $(u,h) \in \mathbf{Z}^d \times \mathbf{Z}_+$, and $s_0 \in \mathbf{Q}$.

As previously in Section 5, next we extend the limit in (6.11) to all rational sites and cells. Fix $y \in \mathbf{Q}^d$, $(x,b) \in \mathbf{Q}^d \times \mathbf{Q}_+$, and $s_0 \in \mathbf{Q}$. Fix an integer m such that $my \in \mathbf{Z}^d$ and $(mx, mb) \in \mathbf{Z}^d \times \mathbf{Z}_+$. For each n, pick $j = j(n)$ so that

$$q(n) \equiv jm \le n < (j+1)m.$$

Then on Γ,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{q(n)} R^{q(n)y}\big(q(n)x, q(n)b\big) \circ \theta_{q(n)s_0} \\
&= \lim_{j\to\infty} \frac{1}{jm} R^{jmy}\big(jmx, jmb\big) \circ \theta_{jms_0} \\
&= \frac{1}{m}\gamma(mx - my, mb) \\
&= \gamma(x-y, b)
\end{aligned} \tag{6.12}$$

by the homogeneity of γ. The inequalities (6.7) and (6.9) can now be repeated, by replacing v, u, h by y, x, b, and by inserting integer parts where appropriate: $[ny]$, $[nx]$, $[nb]$.

We conclude that $\overline{P}(\Gamma) = 1$ for the event Γ defined as follows:

Step 4. The requirements of *Step 2* hold on Γ, and also

$$\lim_{n\to\infty} \frac{1}{n} R^{[ny]}\big([nx], [nb]\big) \circ \theta_{ns_0} = \gamma(x-y, b) \tag{6.13}$$

for all $y \in \mathbf{Q}^d$, $(x,b) \in \mathbf{Q}^d \times \mathbf{Q}_+$, and rational s_0.

The last extension is to arbitrary $y \in \mathbf{R}^d$, $(x,b) \in \mathbf{R}^d \times \mathbf{R}_+$, and real s. Let $\varepsilon > 0$ be rational, and pick rational $\delta \in (0, \varepsilon/2)$ so that also $\delta < \delta_0(\varepsilon/2)$ [$\delta_0(\varepsilon/2)$ as defined in (6.5)]. Pick rational y_0, x_0, b_0, s_0 so that

$$|y_0 - y| + |x_0 - x| + |b_0 - b| + |s_0 - s| \le \delta/2\,.$$

Once more we repeat the reasoning that justified (6.7) and (6.9) to write

$$\begin{aligned}
&R^{[ny_0]}\big([nx_0], [nb_0]\big) \circ \theta_{n(s_0-\varepsilon)} \\
&\le R^{[ny_0]}\big([nx], [nb]\big) \circ \theta_{n(s_0-\varepsilon)} + 2\delta n \\
&\le n(s - s_0 + \varepsilon) + R^{[ny]}\big([nx], [nb]\big) \circ \theta_{ns} + 2\delta n\,,
\end{aligned}$$

and

$$\begin{aligned}
&R^{[ny]}\big([nx],[nb]\big)\circ\theta_{ns}\\
&\le n(s_0-s+\varepsilon)+R^{[ny_0]}\big([nx],[nb]\big)\circ\theta_{n(s_0+\varepsilon)}\\
&\le n(s_0-s+\varepsilon)+R^{[ny_0]}\big([nx_0],[nb_0]\big)\circ\theta_{n(s_0+\varepsilon)}+2\delta n\,.
\end{aligned}$$

Let $n\to\infty$ to get

$$\begin{aligned}
&\gamma(x_0-y_0,b_0)-2\delta-(s-s_0+\varepsilon)\\
&\le \liminf_{n\to\infty}\frac{1}{n}R^{[ny]}([nx],[nb])\circ\theta_{ns}\le \limsup_{n\to\infty}\frac{1}{n}R^{[ny]}([nx],[nb])\circ\theta_{ns}\\
&\le \gamma(x_0-y_0,b_0)+2\delta+(s_0-s+\varepsilon).
\end{aligned}$$

As we let y_0,x_0,b_0,s_0 approach y,x,b,s we can take $\delta,\varepsilon\to 0$. This proves that the limit (6.2) holds on the event Γ, and concludes to proof of Theorem 4.

7. The hydrodynamic limit

In this section we prove Theorem 2. First we work under Assumption A. The processes σ^n are constructed according to the description of Section 3 on a probability space $(\Omega,\mathcal{F},P)$ on which are defined the Poisson jump time processes $\mathcal{T}=\{\mathcal{T}^v\}$ and, statistically independently of $\mathcal{T}$, the sequence of initial interfaces $\{\sigma^n(0)\}$. All processes σ^n use one and the same realization of the Poisson jump time processes $\{\mathcal{T}^v\}$. By Corollary 3.1 there is a single version of the family of ballistic deposition processes $\{Z^v\}$ grown from seeds as defined by (3.6) that satisfy

$$\sigma^n_u(t)=\sup_{v\in\mathbf{Z}^d}\{\sigma^n_v(0)+Z^v_u(t)\} \tag{7.1}$$

for all $n\in\mathbf{N}$, $u\in\mathbf{Z}^d$ and $t\ge 0$, almost surely. Recall that inside the braces the correct convention is $\infty+(-\infty)=-\infty$. The goal is to prove that on some event of full probability, for all $x\in\mathbf{R}^d$ and $t>0$,

$$\lim_{n\to\infty}\frac{1}{n}\sigma^n_{[nx]}(nt)=\psi(x,t)\,. \tag{7.2}$$

For technical reasons we extend the Poisson processes to the entire real line $(-\infty,\infty)$ as was done in Section 6. Let $(\Omega^0,\mathcal{F}^0,P^0)$ be the probability space of the Poisson processes on $(-\infty,0]$, and consider the product space

$$\big(\overline{\Omega},\overline{\mathcal{F}},\overline{P}\big)=(\Omega^0\times\Omega,\mathcal{F}^0\otimes\mathcal{F},P^0\otimes P)\,.$$

A sample point of the product space $\overline{\Omega}$ is $\overline{\omega} = (\omega^0, \omega)$, where $\omega \in \Omega$ represents a realization of $\big(\mathcal{T}, \{\sigma^n(0)\}\big)$ as above, while $\omega^0 \in \Omega^0$ represents a realization of the Poisson processes on the negative time line $(-\infty, 0]$.

If we can prove (7.2) in the product space $\overline{\Omega}$, it follows for the original space Ω too. For suppose $\Gamma \subseteq \overline{\Omega}$ is an event such that $\overline{P}(\Gamma) = 1$ and (7.2) holds on Γ. Pick $\omega^0 \in \Omega^0$ such that the ω^0-section

$$\Gamma_{\omega^0} = \{\omega \in \Omega : (\omega^0, \omega) \in \Gamma\}$$

satisfies $P(\Gamma_{\omega^0}) = 1$. Then (7.2) holds on the full-probability event Γ_{ω^0} because (7.2) depends on ω only, and not on ω^0. So for the remainder of this section assume that all Poisson jump time processes are defined for all real times, and we are on the event of full probability where Assumption A and Theorem 4 from Section 6 are valid.

Recall that ψ is defined for $(x,t) \in \mathbf{R}^d \times (0,\infty)$ by

$$\psi(x,t) = \sup_{y \in x + t\cdot \operatorname{int} B_0} \big\{ \psi_0(y) + tg\big((x-y)/t\big) \big\}\,, \tag{7.3}$$

and on $\mathbf{R}^d \times \{0\}$ set $\psi(x,0) = \psi_0(x)$. This first lemma is a consequence of the continuity and boundedness of g on $\operatorname{int} B_0$.

Lemma 7.1. *Assume ψ_0 is a continuous $[-\infty, +\infty]$-valued function on $\mathbf{R}^d$. Then ψ is a continuous $[-\infty, +\infty]$-valued function on $\mathbf{R}^d \times [0,\infty)$.*

Proof. Let $(x_n, t_n) \to (x,t)$ in $\mathbf{R}^d \times (0,\infty)$. We first argue

$$\liminf_{n\to\infty} \psi(x_n, t_n) \ge \psi(x,t)\,. \tag{7.4}$$

Suppose $\psi(x,t) > -\infty$ [otherwise (7.4) holds trivially]. Let $M < \psi(x,t)$, and pick $y \in x + t \cdot \operatorname{int} B_0$ so that

$$\psi_0(y) + tg\big((x-y)/t\big) > M\,.$$

Since $y \in x_n + t_n \cdot \operatorname{int} B_0$ for large enough n,

$$\begin{aligned}
\liminf_{n\to\infty} \psi(x_n, t_n) &\ge \liminf_{n\to\infty} \left\{ \psi_0(y) + t_n g\left(\frac{x_n - y}{t_n} \right) \right\} \\
&= \psi_0(y) + tg\big((x-y)/t\big) \\
&\ge M\,.
\end{aligned}$$

The equality above follows from the continuity of g on $\operatorname{int} B_0$. This proves (7.4).

To show

$$M_0 \equiv \limsup_{n\to\infty} \psi(x_n, t_n) \le \psi(x,t)\,, \tag{7.5}$$

pick a subsequence n_j so that $\lim_{j\to\infty} \psi(x_{n_j}, t_{n_j}) = M_0$. We may assume $M_0 > -\infty$ and $\psi(x,t) < \infty$. Let $M < M_0$, and for large enough j find $y_j \in x_{n_j} + t_{n_j} \cdot \operatorname{int} B_0$ so that

$$\psi_0(y_j) + t_{n_j} g\big((x_{n_j} - y_j)/t_{n_j}\big) > M\,.$$

Since B_0 is compact and $(x_{n_j}, t_{n_j}) \to (x,t)$, we may pass to a further subsequence so that $y_j \to \overline{y}$. Define

$$y'_j = x - \frac{t}{t_{n_j}}(x_{n_j} - y_j).$$

Then $y'_j \to \overline{y}$, $(x_{n_j} - y_j)/t_{n_j} = (x - y'_j)/t$, and $y'_j \in x + t \cdot \operatorname{int} B_0$. We wish to argue that, for large enough j,

$$-\infty < \psi_0(y_j)\,,\ \psi_0(y'_j) < \infty\,. \tag{7.6}$$

By continuity of ψ_0, this will follow from showing that $-\infty < \psi_0(\overline{y}) < \infty$. We have

$$\psi_0(\overline{y}) = \lim_{j\to\infty} \psi_0(y'_j) \le \psi(x,t) < \infty\,,$$

and

$$\psi_0(\overline{y}) = \lim_{j\to\infty} \psi_0(y_j) \ge M - t\|g\|_\infty > -\infty\,.$$

(7.6) is verified. Now for j large enough,

$$\begin{aligned} M &\le \psi_0(y_j) + t_{n_j} g\big((x_{n_j} - y_j)/t_{n_j}\big) \\ &= \psi_0(y'_j) + tg\big((x - y'_j)/t\big) + (t_{n_j} - t) g\big((x - y'_j)/t\big) \\ &\qquad + \psi_0(y_j) - \psi_0(y'_j) \\ &\le \psi(x,t) + |t_{n_j} - t| \cdot \|g\|_\infty + \psi_0(y_j) - \psi_0(y'_j)\,. \end{aligned}$$

(7.6) was needed to have the difference $\psi_0(y_j) - \psi_0(y'_j)$ defined and convergent to 0. Letting $j \nearrow \infty$ and $M \nearrow M_0$ gives (7.5). This proves continuity on $\mathbf{R}^d \times (0,\infty)$. We omit the similar but shorter argument for the case $(x_n, t_n) \to (x, 0)$. $\square$

One half of the goal (7.2) follows immediately. For arbitrary $y \in x + t \cdot \operatorname{int} B_0$ such that $y \in Y_0$, set $u = [nx]$, $v = [ny]$ and replace t by nt in (7.1), and use assumption (2.5) and Theorem 4 from Section 6 to get

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{n} \sigma^n_{[nx]}(nt) \\ &\ge \liminf_{n\to\infty} \left\{ \frac{1}{n} \sigma^n_{[ny]}(0) + \frac{1}{n} Z^{[ny]}_{[nx]}(nt) \right\} \\ &= \psi_0(y) + tg\big((x-y)/t\big)\,. \end{aligned} \tag{7.7}$$

Note that even though the random variable inside the braces may equal $\infty + (-\infty)$ for finitely many n, eventually $n^{-1} Z_{[nx]}^{[ny]}(nt)$ is finite because its limit $tg\big((x-y)/t\big)$ is finite. For each fixed (x,t) take supremum over these admissible y's to get, by the continuity of the functions involved and by the denseness of Y_0,

$$\liminf_{n\to\infty} \frac{1}{n}\sigma_{[nx]}^n(nt) \geq \psi(x,t). \tag{7.8}$$

(7.8) holds simultaneously for all (x,t) outside a single event of zero probability.

The converse

$$\limsup_{n\to\infty} \frac{1}{n}\sigma_{[nx]}^n(nt) \leq \psi(x,t) \tag{7.9}$$

is where the work is. First we reduce the problem to rational (x,t). Suppose (7.9) holds almost surely for all $(x,t) \in \mathbf{Q}^d \times \big[\mathbf{Q} \cap (0,\infty)\big]$. For rational (x,t) and rational $s > 0$, consider the event

$$D_{n,s}(x,t) = \big\{\sigma_v^n(nt) \leq \sigma_{[nx]}^n(nt+ns) \text{ for all sites } v \\ \text{such that } |v - [nx]| \leq ns/2\big\}.$$

To bound the probability of the complement $D_{n,s}^c(x,t)$, fix n, (x,t), s, and v such that $|v-[nx]| \leq ns/2$. Set $k = \sigma_v^n(nt)$. For times $r \geq nt$, define a new process ρ by $\rho(r) = k + Z^v(r-nt)\circ\theta_{nt}$. By the monotonicity lemma 3.2 $\rho(nt+ns) \leq \sigma^n(nt+ns)$, so $\sigma_{[nx]}^n(nt+ns) < k$ implies that $\rho_{[nx]}(nt+ns) < k$ which in turn is equivalent to $Z_{[nx]}^v(ns)\circ\theta_{nt} = -\infty$. We can estimate as follows:

$$\begin{aligned} P\big(D_{n,s}^c(x,t)\big) &\leq \sum_{v:|v-[nx]|\leq ns/2} P\big(\sigma_v^n(nt) > \sigma_{[nx]}^n(nt+ns)\big) \\ &\leq \sum_{v:|v-[nx]|\leq ns/2} P\big(Z_{[nx]}^v(ns) = -\infty\big) \\ &\leq C_1 n^d P\big(S_{[ns/2]}^1 > ns\big) \\ &\leq C_1 n^d \exp(-C_2 n). \end{aligned}$$

We used the fact that the time for process Z^v to cover cell $([nx],0)$, starting from the seed in cell $(v,0)$, is stochastically dominated by $S_{|v-[nx]|}^1$. The estimate for $P\big(D_{n,s}^c(x,t)\big)$ and Borel-Cantelli imply that, with probability 1, for any rational (x,t) and $s>0$, the event $D_{n,s}(x,t)$ happens for all large enough n.

Given now arbitrary (x_0,t_0), pick rational (x,t) and $s>0$ so that $|x-x_0| < s/4$ and $t_0 < t$. Then eventually $\sigma_{[nx_0]}^n(nt) \leq \sigma_{[nx]}^n(nt+ns)$, while by monotonicity $\sigma_{[nx_0]}^n(nt_0) \leq \sigma_{[nx_0]}^n(nt)$. We get

$$\limsup_{n\to\infty} \frac{1}{n}\sigma_{[nx_0]}^n(nt_0) \leq \psi(x, t+s).$$

Let $t+s \searrow t_0$ and $x \to x_0$, use the continuity Lemma 7.1, and conclude that now (7.9) holds for all (x,t) outside a single exceptional event of zero probability.

It remains to prove that (7.9) holds a.s. for a fixed rational (x,t). Pick rational $0 < s_1 < s_0$, with the intention that $s_0 \searrow 0$ in the end. Set $B_1 = (t+s_1)\cdot B_0$. Define

$$\xi_n = \max_{v\in[nx]+nB_1} \{\sigma^n_v(0) + Z^v_{[nx]}(nt)\}\,. \tag{7.10}$$

Lemma 7.2. *With probability 1, $\sigma^n_{[nx]}(nt) = \xi_n$ for all large enough n.*

Proof. One way to guarantee the equality $\sigma^n_{[nx]}(nt) = \xi_n$ is to require that $Z^v_{[nx]}(nt) = -\infty$ for all $v \notin [nx]+nB_1$. By Lemma 4.1, distributionwise this is equivalent to requiring that $[nx] \notin \mathcal{B}^v(nt)$ for all $v \notin [nx]+nB_1$, where $\mathcal{B}^v$ is a first-passage percolation cluster starting from a seed at site v, defined as in (2.2) in terms of Z^v. Switching to complements,

$$\begin{aligned}
&P\big(\sigma^n_{[nx]}(nt) \neq \xi_n\big)\\
&\le P\big(\ [nx] \in \mathcal{B}^v(nt) \text{ for some } v \notin [nx]+nB_1\ \big)\\
&= P\big(\ 0 \in \mathcal{B}^v(nt) \text{ for some } v \notin nB_1\ \big)\\
&\le \sum_{v\notin nB_1} P\big(0 \in \mathcal{B}^v(nt)\big)\\
&= \sum_{v\notin nB_1} P\big(T(0,v) \le nt\big)\\
&\le C_1 \exp(-C_2 n)
\end{aligned}$$

for constants $C_i = C_i(t,s_1)$, by Corollary 4.1. The conclusion now follows from Borel-Cantelli. $\square$

Let $\{V_i : 1 \le i \le m\}$ be a collection of closed neighborhoods whose interiors cover the compact set $x + B_1$, and such that each

$$V_i \subseteq y_i + (s_0/2)B_0 \tag{7.11}$$

for any $y_i \in V_i$, each V_i lies inside $x + (t+s_0)\cdot \operatorname{int} B_0$, and satisfies assumption (2.6). Since the interiors cover $x + B_1$, we have

$$[nx] + nB_1 \subseteq \bigcup_{i=1}^m nV_i \tag{7.12}$$

for large enough n. Pick $y_i \in V_i$ such that

$$g\left(\frac{x-y_i}{t+s_0}\right) \le \inf_{y\in V_i} g\left(\frac{x-y}{t+s_0}\right) + s_0. \tag{7.13}$$

By (7.11), Corollary 4.2, and Borel-Cantelli, the following holds with probability 1: for large enough n,

$$Z_v^{[ny_i]}(ns_0) \circ \theta_{-ns_0} \geq 0 \text{ for all } v \in nV_i, \tag{7.14}$$

for all $i = 1, \ldots, m$. In words: Start ballistic deposition processes from seeds in cells $([ny_i], 0)$ at Poisson process time $-ns_0$. If n is large enough, at Poisson process time 0 each of these processes has grown sufficiently to cover its piece nV_i. Henceforth assume that we are on this event of full probability, and that n is large enough for (7.14) to hold.

If we define new processes $\widetilde{Z}_u^{[ny_i]}(s) = Z_u^{[ny_i]}(s + ns_0) \circ \theta_{-ns_0}$, (7.14) gives the inequality

$$Z_u^v(s) \leq \widetilde{Z}_u^{[ny_i]}(s) \text{ for all } v \in nV_i \text{ and } u \in \mathbf{Z}^d \tag{7.15}$$

at time $s = 0$, for all $i = 1, \ldots, m$. The monotonicity Lemma 3.2 then ensures that (7.15) holds at all times $s \geq 0$, and we get

$$\begin{aligned} \frac{1}{n}\xi_n &= \max_{v \in [nx] + nB_1} \left\{ \frac{1}{n}\sigma_v^n(0) + \frac{1}{n} Z_{[nx]}^v(nt) \right\} \\ &\leq \max_{1 \leq i \leq m} \left\{ \frac{1}{n} \cdot \max_{v \in nV_i} \sigma_v^n(0) + \frac{1}{n} Z_{[nx]}^{[ny_i]}(nt + ns_0) \circ \theta_{-ns_0} \right\}. \end{aligned} \tag{7.16}$$

Now let $n \to \infty$, apply assumption (2.6), Theorem 4, and the choice (7.13) of y_i. The limit in Theorem 4 can be taken because $y_i \in V_i \subseteq x + (t + s_0) \cdot \operatorname{int} B_0$.

$$\begin{aligned} \limsup_{n \to \infty} \frac{1}{n}\xi_n &\leq \max_{1 \leq i \leq m} \left\{ \sup_{y \in V_i} \psi_0(y) + (t + s_0) g\left(\frac{x - y_i}{t + s_0} \right) \right\} \\ &\leq \sup_{y \in x + (t+s_0) \cdot \operatorname{int} B_0} \left\{ \psi_0(y) + (t + s_0) g\left(\frac{x - y}{t + s_0} \right) \right\} + s_0 \\ &= \psi(x, t + s_0) + s_0 \,. \end{aligned} \tag{7.17}$$

The argument can be repeated for arbitrarily small $s_0 > 0$. Let $s_0 \searrow 0$, use the continuity of ψ (Lemma 7.1), and then Lemma 7.2 to conclude that (7.9) holds a.s. The strong law of Theorem 1 under Assumption A is thereby proved.

Under Assumption B, inequality (7.7), Lemma 7.2, and (7.16)–(7.17) give, almost surely,

$$M \leq \liminf_{n \to \infty} \frac{1}{n} \sigma_{[nx]}^n(nt) \leq \limsup_{n \to \infty} \frac{1}{n} \sigma_{[nx]}^n(nt) \leq \psi(x, t + s_0) + s_0$$

for any $M < \psi(x, t)$ and any $s_0 > 0$. This proves the statement in Theorem 2 under Assumption B. The argument is the same for the weak law under Assumption C.

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